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International Journal of General Systems
2007, 1–29, iFirst article

Do exact shapes of fuzzy sets matter?

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(Received 16 October 2006; revised 20 February 2007; in final form 2 March 2007)

An often raised objection to fuzzy logic can be articulated as follows. Fuzzy sets are intended to represent vague collections of objects. On the one hand, boundaries of such collections are inherently imprecise. On the other hand, fuzzy sets are defined by membership degrees which are precise. There is, therefore, an obvious discrepancy between the imprecision of a collection and the precision of a fuzzy set representing the collection. The objection leads to a question posed in the title of this paper: do exact shapes of fuzzy sets matter? That is, do the exact values of membership degrees of fuzzy sets matter? This is a fundamental question, particularly when fuzzy sets are supplied as an input to further processing.

This paper presents a particular answer to this question. We focus on the case when the fuzzy sets in question are part of the input I from which the output O is produced in a way which can be described by logical formulas. This case covers several widely used fuzzy logical models including extension principle, products of fuzzy relations, Zadeh's compositional rule of inference, fuzzy automata, and properties of fuzzy relations. We present formulas which say how similar is the original output O to a new output O' when the original input I is replaced by a new input I'.

The presented formulas provide us with an analytical tool which enables us to answer the question of whether (and to what extent) the exact shapes of fuzzy sets in a particular fuzzy logical model matter. We present application of our formulas to selected topics including the above-mentioned extension principle, fuzzy relational products, Zadeh's compositional rule of inference, fuzzy automata, and properties of fuzzy relations. Based on the analysis, we argue that in several applications of fuzzy logic, the exact shapes of fuzzy sets do not really matter.

Keywords: *Fuzzy set; Membership degree; Similarity; Sensitivity; Predicate fuzzy logic*

1. The question and our answer in brief

1.1 Imprecise boundaries vs. precise membership degrees: really a problem?

Fuzzy sets and fuzzy relations (Zadeh 1965) are intended to represent meaning of natural language expressions. More often than not, natural language expressions denote vague collections of objects and vague relationships between objects like "tall men", "high temperature", "is similar to", etc. Vague collections are said to have imprecise boundaries, contrary to crisp collections, such as "prime number", for which the boundaries are clearly defined. The imprecision of boundaries of vague collections contrasts the precision of the

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International Journal of General Systems
ISSN 0308-1079 print/ISSN 1563-5104 online © 2007 Taylor & Francis
http://www.tandf.co.uk/journals
DOI: 10.1080/03081070701359167

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membership degrees of fuzzy sets. Namely, a fuzzy set in a universe U is a mapping A assigning membership degrees $A(u) \in [0,1]$ to elements $u \in U$. The boundary of a fuzzy set is thus graded rather than crisp but, nevertheless, precise. Hence the problem of imprecise boundaries vs. precise membership functions. Note that fuzzy sets of type 2, see (Klir and Yuan 1995), just move this problem one step farther rather than removing it.

The problem of imprecise boundaries vs. precise membership functions is an example of problems encountered when dealing with mathematization of reality. Mathematical objects which serve as models of pieces of reality are always precise and are never fully fitting reality. A similar problem to that of imprecise boundaries vs. precise membership functions is that of subjective beliefs vs. precise probability distribution functions in subjective probability, or that of inherently imprecise boundaries of real-world objects such as a body of an aircraft vs. precise mathematical models used for simulation purposes. Therefore, the problem of imprecise boundaries vs. precise membership functions is not conceptually new. Rather, it is a particular example of a general phenomenon commonly encountered in mathematization of reality, namely, that of complexity and imperfection of real world vs. simplicity and precision of mathematical models.

1.2 Coming to our question: do exact shapes of fuzzy sets matter? And to what extent?

People successfully communicate knowledge using natural language which inherently involves vague collections like “tall men”, “high temperature”, etc. Needless to say, when people communicate, they do not refer to membership degrees. However, when we use fuzzy sets to represent vague collections, membership degrees are crucial. Namely, a fuzzy set A is, by definition, given by the membership degrees which A assigns to objects from the universe. On the other hand, a common experience shared by many practitioners says that the exact values of membership degrees, i.e. exact shapes of fuzzy sets, do not really matter. That is, the experience says, it is not of crucial importance whether we set a membership degree $A(x)$ of object x in fuzzy set A to, say, $A(x) = 0.7$ or $A(x) = 0.71$. Without a proper analysis, the experience-based argument supporting a claim that exact shapes of fuzzy sets do not matter, is not quite convincing and needs to be taken with caution. An analysis we are calling for should reveal the consequences of having $A(x) = 0.7$ instead of $A(x) = 0.71$, preferably in mathematical terms which bear simple and clear meaning.

To perform such an analysis and to answer the question posed in the title of this paper is our main purpose. In particular, we are looking for an answer which helps us see to what extent the exact shapes of fuzzy sets matter.

1.3 Our answer in brief

The way we attempt to answer the question from the title of this paper can be described as follows. We consider situations where there are input fuzzy sets or fuzzy relations $I_1, \ldots, I_n$ and a mapping F which assigns to $I_1, \ldots, I_n$ a new fuzzy set or fuzzy relation O. That is, we consider a mapping

$$F : \langle I_1, \ldots, I_n \rangle \mapsto O = F(I_1, \ldots, I_n). \tag{1}$$

F represents a process (reasoning, computation) using which we derive O from $I_1, \ldots, I_n$. Our approach does not apply to arbitrary F. We consider a particular but quite general form of F.

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Namely, we assume that F can be described using logical formulas. Such kind of processing is very common in fuzzy logic modeling. For instance, it includes the following well-known examples:

Zadeh's extension principle (Zadeh 1975), where $F:[0,1]^X \longrightarrow [0,1]^Y$ is a function which results from a function $f:X \longrightarrow Y$ using Zadeh's extension principle, see also (Klir and Yuan 2002).
Fuzzy relational products, including circle-, triangular-, and square-products, see (Kohout and Kim 2002), where for fuzzy relations $R \in [0,1]^{X \times Y}$ and $S \in [0,1]^{Y \times Z}$, $F(R,S)$ is the product (composition) of R and S.
Zadeh's compositional rule of inference (CRI), see e.g. (Klir and Yuan 2002), where for an input fuzzy set $A \in [0,1]^X$ and a fuzzy relation $R \in [0,1]^{X \times Y}$, $F(A,R) \in [0,1]^Y$ is the corresponding fuzzy set derived from A and R by CRI. For the sake of illustration, let us note that a logical formula corresponding to F in case of Zadeh's compositional rule of inference is $(\exists x)(r_A(x) \otimes r_R(x,y))$ where r_A and r_R are relation symbols corresponding to fuzzy set A and fuzzy relation R, respectively.
Fuzzy automata, where for given fuzzy sets Q_I and Q_F of initial and final states, and a fuzzy transfer relation δ, $F(Q_I,Q_F,\delta)$ is a fuzzy set of words accepted by the fuzzy automaton given by Q_I, Q_F, and δ.

Next, we consider collections of input fuzzy sets and the corresponding output fuzzy sets, namely

$$\text{original input fuzzy sets } \langle I_1, \ldots, I_n \rangle \mapsto \text{original output } O \tag{2}$$

$$\text{modified input fuzzy sets } \langle I'_1, \ldots, I'_n \rangle \mapsto \text{new output } O', \tag{3}$$

and we ask the following question:

$$\text{To what extent are } O \text{ and } O' \text{ similar in terms of similarity of } I_1, \ldots, I_n \text{ to } I'_1, \ldots, I'_n? \tag{4}$$

Then, we use a particular way to measure similarity between fuzzy sets and fuzzy relations, which enables us to consider degrees of similarity $(I_1 \approx I'_1) \in [0,1], \ldots, (I_n \approx I'_n) \in [0,1]$, and $(O \approx O') \in [0,1]$, to which I_1 is similar to $I'_1, \ldots$, and to which O is similar to O'. Our answer to (4) is then presented in a form of estimation formulas such as

$$(I_1 \approx I'_1) \leq (O \approx O') \tag{5}$$

in case of F with a single input or

$$(I_1 \approx I'_1) \otimes \cdots \otimes (I_n \approx I'_n) \leq (O \approx O'). \tag{6}$$

in case of F with n inputs. In general, however, our estimation formulas look like

$$(I_1 \approx I'_1)^{k_1} \otimes \cdots \otimes (I_n \approx I'_n)^{k_n} \leq (O \approx O'), \tag{7}$$

where $k_1, \ldots, k_n$ are non-negative integers which depend on (and can be efficiently determined from) the logical formula corresponding to F. Here, $\otimes$ is a "fuzzy conjunction" and $(I_i \approx I'_i)^{k_i}$ stands for $(I_i \approx I'_i) \otimes \cdots \otimes (I_i \approx I'_i)$, k_i-times. In fact, our formulas are a bit more versatile but we omit details at this point for simplicity. Clearly, (7) provides us with an easy-to-read numerical estimation. As an example, suppose we have F with two inputs $(n = 2)$, $k_1 = 1$ and $k_2 = 2$, and use Łukasiweicz conjunction $\otimes$. Suppose furthermore that

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instead of input fuzzy sets I_1 and I_2, we use slightly different fuzzy sets I_1' and I_2' with degrees of similarity being $(I_1 \approx I_1') = 0.9$ and $(I_2 \approx I_2') = 0.95$. Then, since $(I_1 \approx I_1') \otimes (I_2 \approx I_2') = 0.9 \otimes (0.95)^2 = 0.9 \otimes 0.95 \otimes 0.95 = 0.8$, (7) says:

$$\text{if } (I_1 \approx I_1') = 0.9 \quad \text{and} \quad (I_2 \approx I_2') = 0.95, \quad \text{then} \quad 0.8 \leq (O \approx O').$$

That is, a degree of similarity of the corresponding output fuzzy sets $O = F(I_1, I_2)$ and $O' = F(I_1', I_2')$ needs to be at least 0.8. This enables us to conclude that small changes in the input fuzzy sets lead to small changes in the output fuzzy sets with (7) bearing the exact meaning of this claim.

Let us remark that our estimation formulas have a natural linguistic description, and thus a natural interpretation. Namely, since for any truth degrees $a,b \in [0,1]$ we have $a \leq b$ iff $a \rightarrow b = 1$ where $\rightarrow$ is a "fuzzy implication", (5) says "if input I_1 is similar to input I_1' then output O is similar to output O'". Moreover, since $\otimes$ is a "fuzzy conjunction", (6) says "if input I_1 is similar to input I_1', and $\ldots$, and input I_n is similar to input I_n', then output O is similar to output O'".

1.4 Do, then, exact shapes really matter?

Suppose input fuzzy sets $I_1, \ldots, I_n$ are being processed using F and that, as a result, we obtain a fuzzy set O. Then, (7) says:

$$\text{Similar input fuzzy sets lead to similar output fuzzy sets.} \tag{8}$$

This can be regarded as saying that exact shapes of fuzzy sets do not matter because if we replace input fuzzy sets $I_1, \ldots, I_n$ by pairwise similar fuzzy sets $I_1', \ldots, I_n'$, we obtain an output fuzzy set O' which is similar to the original output fuzzy set O.

One needs, however, to keep in mind the numerical meaning of (8), and this is given by (7). (7) gives an estimation of similarity of the output fuzzy sets in terms of similarity of input fuzzy sets. Since larger coefficients k_i lead to smaller left-hand side of (7), one can conclude that the smaller the coefficients k_i, the less sensitive the mapping F is to changes in the input fuzzy sets. As we will see later, all the coefficients k_i in the above-mentioned examples (extension principle, relational composition, compositional rule of inference, fuzzy automata) happen to equal 1. Therefore, a concise version of our answer is:

Answer: If fuzzy sets enter processing F which can be described by logical formulas, then similar input fuzzy sets lead to similar outputs. Therefore, in such case, which is quite common in fuzzy logic modeling, exact shapes of fuzzy sets do not matter. The dependence of similarity of outputs on similarity of inputs is described by (7).

1.5 Limits of our approach

We assume that the input fuzzy sets enter a processing which yields a fuzzy set as an output. Our analysis focuses on how the output changes when the input changes. The main limits are:

- We assume that the processing can be described using logical formulas. This is very common in fuzzy logic modeling, though not always explicitly articulated. Nevertheless, our results do not apply to a processing which cannot be described by logical formulas as understood in our approach.
- We use a particular way to assess similarity between fuzzy sets and fuzzy relations. For other measures of similarity, which might well be reasonable, our approach does not apply.

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1.6 Organization of details presented below

In the rest of the paper, we present details of our analysis. Section 2 contains preliminaries from fuzzy sets and fuzzy logic. Note that we work in a general setting and use complete residuated lattices as our structures of truth degrees. This leaves several important types of structures of truth degrees a particular case, the most important of them being the unit interval [0,1] equipped with an arbitrary left-continuous *t*-norm as a truth function of a "fuzzy conjunction" and the corresponding residuum as a truth function of "fuzzy implication". Section 3 presents our results. Examples of application of our results to selected examples of fuzzy logic modeling are the content of Section 4. Section 5 summarizes our paper. Proofs and auxiliary results are presented in Appendix.

Note that the technical results on which our analysis is based were presented in a bit more general framework in Belohlavek (2002). In addition to presenting the technical results in a more particular and thus more comprehensible form, we emphasize applicational aspects of these results and provide several illustrative examples in the present paper. Most importantly, however, we aim to answer the question posed in the title of our paper.

2. Preliminaries

2.1 Structures of truth degrees

When working in a "fuzzy setting", one needs to select an appropriate structure $\mathbf{L}$ of truth degrees, i.e. a set L of truth degrees such as $L = [0,1]$ equipped with "fuzzy logical connectives" such as the Łukasiewicz conjunction $\otimes$ defined by $a \otimes b = \max(0, a + b - 1)$ as a "fuzzy conjunction". Our approach covers a broad spectrum of structures of truth degrees. Rather than working with a single particular structure of truth degrees, we allow any structure of truth degrees satisfying a predefined set of properties which are natural from a logical point of view. The structures of truth degrees satisfying the properties we require are called complete residuated lattices. Complete residuated lattices are commonly used structures of truth degrres, both in fuzzy logic applications and in foundations of fuzzy logic. We assume basic familiarity with their properties, see (Goguen 1967, Höhle 1996, Hájek 1998, Belohlavek 2002). However, nothing essential is lost if the reader fixes a particular complete residuated lattice $\mathbf{L}$ such as the one given by Łukasiewicz operations $a \otimes b = \max(0, a + b - 1)$ and $a \rightarrow b = \min(1, 1 - a + b)$ on [0,1]. That is, although the results we present are valid for every particular complete residuated lattice $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$, the reader can always think of $\mathbf{L}$ the following way: $L = [0,1]$, $a \wedge b = \min(a, b)$, $a \vee b = \max(a, b)$, $a \otimes b = \max(0, a + b - 1)$, $a \rightarrow b = \min(1, 1 - a + b)$ for $a,b \in [0,1]$.

Definition 1. A (complete) residuated lattice is an algebra $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ such that $\langle L, \wedge, \vee, 0, 1\rangle$ is a (complete) lattice with the least element 0 and the greatest element 1, $\langle L, \otimes, 1\rangle$ is a commutative monoid (i.e. $\otimes$ is a commutative and associative binary operation on L satisfying $a \otimes 1 = a$), and for $\otimes$ and $\rightarrow$ we have

$$a \otimes b \leq c \quad \text{if and only if} \quad a \leq b \rightarrow c \tag{9}$$

for each a, b, $c \in L$.

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The set L plays the role of a set of truth degrees, 0 and 1 representing full falsity and full truth, respectively. Operations on L correspond to logical connectives: $\otimes$ and $\rightarrow$ are truth functions of "fuzzy conjunction" and "fuzzy implication", respectively. $\bigvee$ and $\bigwedge$ serve for expressing semantics of the existential quantifier $\exists$ and the general quantifier $\forall$, respectively. In every residuated lattice, one can introduce a binary operation $\leftrightarrow$ by

$$a \leftrightarrow b = (a \rightarrow b) \wedge (b \rightarrow a). \tag{10}$$

$\leftrightarrow$ is called a biresiduum and serves as a truth function of "fuzzy equivalence". All properties of complete residuated lattices used in the sequel are well-known and can be found in the above mentioned literature. Note that particular types of residuated lattices include Boolean algebras, Heyting algebras, algebras of Girard's linear logic, MV-algebras, Gödel algebras, product algebras, and more generally, BL-algebras, see (Höhle 1996, Hájek 1998). These special residuated lattices can be distinguished by identities which correspond to logical requirements. For example, Boolean algebras are BL-algebras satisfying $a = (a \rightarrow 0) \rightarrow 0$ which is an identity corresponding to the law of double negation (note that $b \rightarrow 0$ is the negation of b).

2.2 Examples of particular structures of truth degrees

A common choice of L is the real unit interval [0,1] or a subchain of [0,1]. It is well-known that $\mathbf{L} = \langle [0,1], \min, \max, \otimes, \rightarrow, 0, 1\rangle$ is a complete residuated lattice if and only if $\otimes$ is a left-continuous t-norm and $\rightarrow$ is defined by $a \rightarrow b = \max\{c | a\otimes c \leq b\}$. Recall that a t-norm is a binary operation on [0,1] which is associative, commutative, monotone, and has 1 as its neutral element, and hence, captures basic intuitive properties of conjunction. A t-norm is called left-continuous if, as a real function, it is left-continuous in both arguments. Most commonly used are continuous t-norms, the basic three of which are the Łukasiewicz t-norm (given by $a\otimes b = \max(a + b - 1, 0)$ with the corresponding residuum $a \rightarrow b = \min(1 - a + b, 1)$), minimum (also called Gödel) t-norm ($a\otimes b = \min(a, b)$, $a \rightarrow b = 1$ if $a \leq b$ and $= b$ otherwise), and product t-norm ($a\otimes b = a\cdot b$, $a \rightarrow b = 1$ if $a \leq b$ and $= b/a$ otherwise). Any finite subchain of [0,1] containing both 0 and 1, equipped with restrictions of the minimum t-norm and its residuum is a complete residuated lattice. Furthermore, the same holds true for any equidistant finite chain $\{0,(1/n),\ldots,((n-1)/n),1\}$ equipped with restrictions of Łukasiewicz operations. The only residuated lattice on the two-element chain $\{0,1\}$ (with $0 < 1$) is, in fact, the two-element Boolean algebra of classical logic, i.e. $\otimes$ and $\rightarrow$ are the classical conjunction and the classical implication. Throughout the paper, $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ denotes a complete residuated lattice.

2.3 Fuzzy sets and fuzzy relations

An **L**-set (or fuzzy set with truth degrees from a complete residuated lattice **L**) in a universe X is any mapping A from X to L (in general, we use "**L**- ..." instead of "fuzzy ..." to make the structure of truth degrees explicit). Thus, for a residuated lattice **L** with $L = [0,1]$, we get the usual notion of a fuzzy set A in a universe X as a mapping A from X to [0,1]. This is how the notion of an **L**-set generalizes the usual notion of a fuzzy set. By L^X we denote the collection of all **L**-sets in X. A truth degree $A(x) \in L$ is interpreted as a truth degree to which

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x belongs to A. An n-ary **L**-relation in a set X is an **L**-set in X^n. For example, a binary relation R in X is a mapping $R{:}X \times X \to L$.

A binary **L**-relation E on X is called an **L**-equivalence if

$$E(x,x) = 1 \quad E(x,y) = E(y,x) \quad E(x,y) \otimes E(y,z) \leq E(x,z)$$

hold true for any $x, y, z \in X$. These conditions are called reflexivity, symmetry, and transitivity, respectively. An **L**-equivalence E is called an **L**-equality if $x = y$ whenever $E(x,y) = 1$.

2.4 Structures for predicate fuzzy logic

In this section, we recall basic concepts of fuzzy predicate logic. A language $\mathcal{J}$ of our logic contains: a non-empty set R of relation symbols, each $r \in R$ with its arity $\sigma(r)$, a (possibly empty) set F of function symbols, each $f \in F$ with its arity $\sigma(f)$; object variables $x_1, x_2, \ldots, x, y, z, \ldots, \xi, \nu, \ldots$ etc.; logical connectives $\wedge, \vee, \otimes, \Rightarrow$; truth constants $\mathbb{0}$ and $\mathbb{1}$ (and possibly others), quantifiers $\forall$ and $\exists$ (universal and existential); auxiliary symbols. Furthermore, R contains a binary relation symbol $\approx$ (equality) which is then handled in a special way. Namely, it is interpreted as a fuzzy equivalence and is required to be compatible with fuzzy relations and functions which serve as interpretations of relation and function symbols from R and F. Such a language $\mathcal{J}$ is also said to be of type $\langle R,F,\sigma\rangle$. Terms and formulas are defined as usual. Terms: each variable is a term; if t_i are terms and $f \in F$ then $f(\ldots,t_i,\ldots)$ is a term. Formulas: truth constants $\mathbb{0}, \mathbb{1}, \ldots,$ are formulas; $r(\ldots, t_i, \ldots)$ are formulas ($r \in R$, t_i terms); if φ and ψ are formulas then $(\varphi \wedge \psi$, $\varphi \vee \psi$, $\varphi \otimes \psi$, $(\varphi \Rightarrow \psi)$, $(\forall x)\varphi$, and $(\exists x)\varphi$ are formulas.

Semantics is defined as usual: an **L**-structure $\mathbf{M} = \langle M, R^{\mathbf{M}}, F^{\mathbf{M}}\rangle$ for $\mathcal{J}$ consists of a non-empty set M (universe), a set $R^{\mathbf{M}} = \{r^{\mathbf{M}}{:}M^n \to L | r \in R, \sigma(r) = n\}$ of **L**-relations, and a set $F^{\mathbf{M}} = \{f^{\mathbf{M}}{:}M^n \to M | f \in F, \sigma(r) = n\}$ of functions such that $\approx^{\mathbf{M}}$ is an **L**-equivalence relation on M and each $r^{\mathbf{M}} \in R^{\mathbf{M}}$ and $f^{\mathbf{M}} \in F^{\mathbf{M}}$ is compatible with $\approx^{\mathbf{M}}$. Compatibility of an n-ary $r^{\mathbf{M}}$ and and n-ary $f^{\mathbf{M}}$ with $\approx^{\mathbf{M}}$ means that

$$(m_1 \approx^{\mathbf{M}} m_1') \otimes \cdots \otimes (m_n \approx^{\mathbf{M}} m_n') \otimes r^{\mathbf{M}}(m_1, \ldots, m_n) \leq r^{\mathbf{M}}(m_1', \ldots, m_n')$$

and

$$(m_1 \approx^{\mathbf{M}} m_1') \otimes \cdots \otimes (m_n \approx^{\mathbf{M}} m_n') \leq f^{\mathbf{M}}(m_1, \ldots, m_n) \approx^{\mathbf{M}} f^{\mathbf{M}}(m_1', \ldots, m_n')$$

for any $m_1, m_1', \ldots, m_n, m_n' \in M$. Note that if $\approx^{\mathbf{M}}$ is interpreted as a similarity, i.e. $m \approx^{\mathbf{M}} m'$ is a degree to which m and m' are similar, then the compatibility of $r^{\mathbf{M}}$ says "if m_i and m_i' are pairwise similar and $m_1, \ldots, m_n$ are related by r then $m_1', \ldots, m_n'$ are related by r as well". Furthermore, the compatibility of $f^{\mathbf{M}}$ says "if m_i and m_i' are pairwise similar then $f^{\mathbf{M}}(m_1, \ldots, m_n)$ and $f^{\mathbf{M}}(m_1', \ldots, m_n')$ are similar". Note that compatibility of $r^{\mathbf{M}}$ may be equivalently expressed by

$$(m_1 \approx^{\mathbf{M}} m_1') \otimes \cdots \otimes (m_n \approx^{\mathbf{M}} m_n') \leq r^{\mathbf{M}}(m_1, \ldots, m_n) \leftrightarrow r^{\mathbf{M}}(m_1', \ldots, m_n').$$

That is, for each relation symbol $r \in R$, $\mathbf{M}$ contains a corresponding **L**-relation $r^{\mathbf{M}}$; for each function symbol $f \in F$, $\mathbf{M}$ contains a corresponding (ordinary) function $f^{\mathbf{M}}$; and all $r^{\mathbf{M}}$'s and $f^{\mathbf{M}}$'s in a natural way respect the underlying similarity $\approx^{\mathbf{M}}$. An $\mathbf{M}$-valuation of object variables is a mapping v assigning an element $v(x) \in M$ to any variable x. If v and v' are valuations, and x a variable we write $v =_x v'$ if for each variable $y \neq x$ we have

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$v(y) = v'(y)$. A value $\|t\|_{\mathbf{M},v}$ of a term t under an **M**-valuation v is defined as follows:

$$\text{atomic terms}: \|x\|_{\mathbf{M},v} = v(x);$$

$$\text{compound terms}: \|f(t_1, \ldots, t_n)\|_{\mathbf{M},v} = f^{\mathbf{M}}(\|t_1\|_{\mathbf{M},v}, \ldots, \|t_n\|_{\mathbf{M},v}).$$

A truth degree $\|\varphi\|_{\mathbf{M},v}$ of a formula φ is defined as follows:

(i) atomic formulas:
$$\|r(t_1,\ldots,t_n)\|_{\mathbf{M},v} = r^{\mathbf{M}}(\|t_1\|_{\mathbf{M},v},\ldots,\|t_n\|_{\mathbf{M},v}), \|\mathbb{0}\|_{\mathbf{M},v} = 0, \|\mathbb{1}\|_{\mathbf{M},v} = 1;$$

(ii) compound formulas:
$$\|\varphi \wedge \psi\|_{\mathbf{M},v} = \|\varphi\|_{\mathbf{M},v} \wedge \|\psi\|_{\mathbf{M},v}, \|\varphi \vee \psi\|_{\mathbf{M},v} = \|\varphi\|_{\mathbf{M},v} \vee \|\psi\|_{\mathbf{M},v},$$
$$\|\varphi \otimes \psi\|_{\mathbf{M},v} = \|\varphi\|_{\mathbf{M},v} \otimes \|\psi\|_{\mathbf{M},v}, \|\varphi \Rightarrow \psi\|_{\mathbf{M},v} = \|\varphi\|_{\mathbf{M},v} \to \|\psi\|_{\mathbf{M},v},$$
$$\|(\forall x)\varphi\|_{\mathbf{M},v} = \bigwedge\{\|\varphi\|_{\mathbf{M},v'} \mid v' =_x v\}, \|(\exists x)\varphi\|_{\mathbf{M},v} = \bigvee\{\|\varphi\|_{\mathbf{M},v'} \mid v' =_x v\}.$$

The foregoing definitions can be easily extended for a many-sorted case. In a many-sorted case, instead of a universe set M, one can have several universe sets with different sorts of elements, e.g. M_1 containing persons, M_2 consisting of companies, M_3 consisting of numbers, etc. In the many-sorted framework, one can consider many-sorted fuzzy relations like "a person often makes deals with a company". See Remark 2 and also (Hájek 1998) for details.

3. The question and our answer in detail

Our approach can be described as follows. The original input fuzzy sets and relations $I_1, \ldots, I_n$ are considered to be fuzzy sets and fuzzy relations $r_1^{\mathbf{M}_1}, \ldots, r_n^{\mathbf{M}_1}$ of an appropriate **L**-structure $\mathbf{M}_1$, cf. (2). The modified input fuzzy sets and relations $I'_1, \ldots, I'_n$ are considered to be fuzzy sets and fuzzy relations $r_1^{\mathbf{M}_2}, \ldots, r_n^{\mathbf{M}_2}$ of a different **L**-structure $\mathbf{M}_2$, cf. (3). $\mathbf{M}_1$ and $\mathbf{M}_2$ have a common universe and a common fuzzy equivalence relation, i.e. M_1 coincides with M_2 and $\approx^{\mathbf{M}_1}$ coincides with $\approx^{\mathbf{M}_2}$. Therefore, we often write just M instead of M_1 or M_2, and $\approx^{\mathbf{M}}$ instead of $\approx^{\mathbf{M}_1}$ or $\approx^{\mathbf{M}_2}$. The output fuzzy sets or fuzzy relations O and O' are then defined by an appropriate formula φ with a free variable x. Formula φ corresponds to the mapping F from (1). For illustration, suppose O and O' are fuzzy sets in M_1. Then, we define O and O' by

$$O(m) = \|\varphi\|_{\mathbf{M}_1,v} \quad \text{and} \quad O'(m) = \|\varphi\|_{\mathbf{M}_2,v}$$

for each m of the universe where v is a valuation which maps the free variable x of φ to m. Next, we define syntactical notions of a degree $|\varphi|_y$ of a variable y in a formula φ, degree $|\varphi|_r$ of a relation symbol r in φ, etc. The degrees $|\varphi|_x$, $|\varphi|_r$, etc. are non-negative integers and serve as the coefficients $k_1, \ldots, k_n$ in our estimation formulas like (7). The scheme of our answer to (4) is presented in figure 1. After that, in Section 3.2, we present our estimation formulas, namely, Theorem 1 and 2, of which (7) is a particular example.

3.1 Basic definitions

In this section, we introduce auxiliary notions we need to formulate our results.

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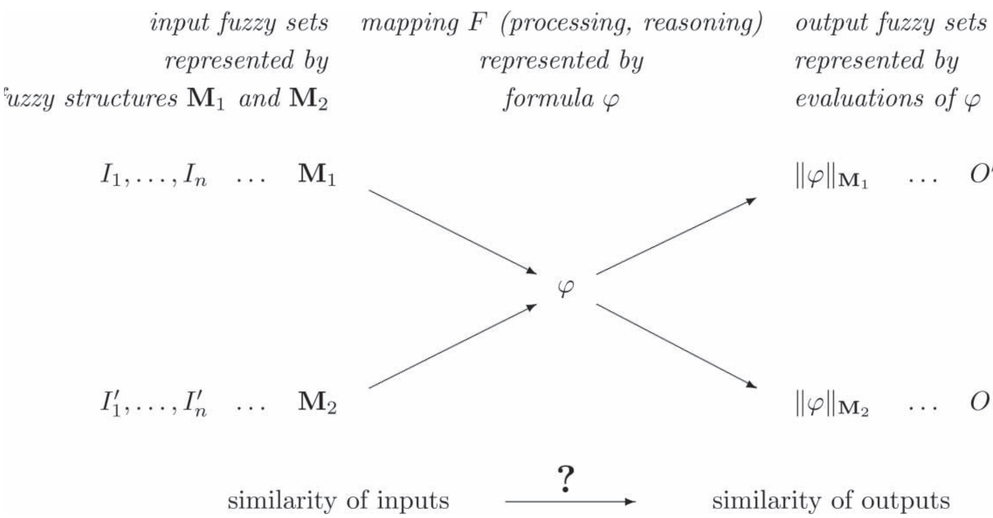

Figure 1. Do similar similar input fuzzy sets I_i and I'_i lead to similar output fuzzy sets O and O'?

Definition 2. For a term t and a variable x we define a degree $|t|_x$ of x in t by

(i) if t is a variable then

$$|t|_x = \begin{cases} 1 & \text{if } t = x \\ 0 & \text{if } t \neq x; \end{cases}$$

(ii) if $t = f(t_1, \ldots, t_n)$ then $|t|_x = |t_1|_x + \cdots + |t_n|_x$.

It is easy to see that $|t|_x$ is the number of occurrences of x in t.

Definition 3. For a term t and a function symbol f we define a degree $|t|_f$ of f in t by

(i) if t is a variable then $|t|_f = 0$;
(ii) if $t = g(t_1, \ldots, t_n)$ then

$$|t|_f = \begin{cases} 1 + |t_1|_f + \cdots + |t_n|_f & \text{if } f = g \\ |t_1|_f + \cdots + |t_n|_f & \text{if } f \neq g; \end{cases}.$$

Again, $|t|_f$ is the number of occurrences of f in t.

Definition 4. For a formula φ and a variable x we define a degree $|\varphi|_x$ of x in φ by

(i) for atomic formulas: for a truth constant $\mathbb{t}$, $|\mathbb{t}|_x = 0$ (in particular, $|\mathbb{0}|_x = 0, |\mathbb{1}|_x = 0$); if $\varphi = s(t_1, \ldots, t_n)$ then

$$|\varphi|_x = \begin{cases} |t_1|_x + \cdots + |t_n|_x & \text{if } s = r \\ 0 & \text{if } s \neq r; \end{cases}$$

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(ii) for compound formulas:

$$|\varphi \wedge \psi|_x = \max(|\varphi|_x, |\psi|_x), \quad |\varphi \vee \psi|_x = \max(|\varphi|_x, |\psi|_x),$$
$$|\varphi \otimes \psi|_x = |\varphi|_x + |\psi|_x, \quad |\varphi \Rightarrow \psi|_x = |\varphi|_x + |\psi|_x,$$
$$|(\forall y)\varphi|_x = |\varphi|_x, \quad |(\exists y)\varphi|_x = |\varphi|_x.$$

DEFINITION 5. For a formula φ and a relation symbol r we define a degree $|\varphi|_r$ of r in φ by

(i) for atomic formulas: for a truth constant $\mathbb{t}$, $|\mathbb{t}|_r = 0$ (in particular, $|\mathbb{0}|_r = 0, |\mathbb{1}|_r = 0$); if $\varphi = s(t_1, \ldots, t_n)$ then

$$|\varphi|_r = \begin{cases} 1 & \text{if } s = r \\ 0 & \text{if } s \neq r; \end{cases}$$

(ii) for compound formulas:

$$|\varphi \wedge \psi|_r = \max(|\varphi|_r, |\psi|_r), \quad |\varphi \vee \psi|_r = \max(|\varphi|_r, |\psi|_r),$$
$$|\varphi \otimes \psi|_r = |\varphi|_r + |\psi|_r, \quad |\varphi \Rightarrow \psi|_r = |\varphi|_r + |\psi|_r,$$
$$|(\forall x)\varphi|_r = |\varphi|_r, \quad |(\exists x)\varphi|_r = |\varphi|_r.$$

DEFINITION 6. For a formula φ and a function symbol f we define a degree $|\varphi|_f$ of f in φ by

(i) for atomic formulas: for a truth constant $\mathbb{t}$, $|\mathbb{t}|_f = 0$ (in particular, $|\mathbb{0}|_f = 0, |\mathbb{1}|_f = 0$); if $\varphi = r(t_1, \ldots, t_n)$ then $|\varphi|_f = |t_1|_f + \cdots + |t_n|_f$.

(ii) for compound formulas:

$$|\varphi \wedge \psi|_f = \max(|\varphi|_f, |\psi|_f), \quad |\varphi \vee \psi|_f = \max(|\varphi|_f, |\psi|_f),$$
$$|\varphi \otimes \psi|_f = |\varphi|_f + |\psi|_f, \quad |\varphi \Rightarrow \psi|_f = |\varphi|_f + |\psi|_f,$$
$$|(\forall x)\varphi|_f = |\varphi|_f, \quad |(\exists x)\varphi|_f = |\varphi|_f.$$

By var(t), we denote, the set of all variables which occur in a term t, i.e.

$$\text{var}(t) = \{x \| t|_x \geq 1\}.$$

By free(φ) we denote the set of all free variables of a formula φ defined in the usual way. We write $\bigotimes_{i=1}^{n} a_i$ to denote $a_1 \otimes \cdots \otimes a_n$.

Furthermore, we make use of the following definition of similarity degrees of fuzzy relations and functions. Given n-ary **L**-relations r and s in M, we define a degree $r \approx s$ of similarity of r and s by

$$(r \approx s) = \bigwedge_{m_1, \ldots, m_n \in M} r(m_1, \ldots, m_n) \leftrightarrow s(m_1, \ldots, m_n). \tag{11}$$

Note that $r \approx s$ is a truth degree of "for any $m_1, \ldots, m_n$: $m_1, \ldots, m_n$ are in r iff $m_1, \ldots, m_n$ are in s". Given n-ary functions $f, g : M^n \to M$ on a set M with an **L**-equivalence $\approx^M$ in M,

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we define a degree $f \approx g$ of similarity of f and g by

$$(f \approx g) = \bigwedge_{m_1, \ldots, m_n \in M} f(m_1, \ldots, m_n) \approx^M g(m_1, \ldots, m_n). \tag{12}$$

Note that $f \approx g$ is a truth degree of "for any $m_1, \ldots, m_n$: $f(m_1, \ldots, m_n)$ is similar to $g(m_1, \ldots, m_n)$".

Note also that we are using symbol $\approx$ in two different meanings. First, $\approx$ is a relational symbol from R; second, $\approx$ denotes a fuzzy relation defined by (11). It is, however, always clear from the context which meaning $\approx$ bears. Note also that $r \approx s$ given by (11) is quite a common way to define a similarity degree of two fuzzy sets or fuzzy relations, see e.g. (Hájek 1998, Gottwald 2001). The following lemma is easy to see, cf. (Hájek 1998, Gottwald 2001, Belohlavek 2002).

LEMMA 1. The **L**-relations defined by (11) and (12) are **L**-equivalences on the sets of all n-ary **L**-relations and n-ary functions on M, respectively. Moreover, (11) is an **L**-equality; (12) is an **L**-equality whenever $\approx^M$ is an **L**-equality.

3.2 Results

In this section, we present our main technical results. Note that we consider two **L**-structures $\mathbf{M}_1$ and $\mathbf{M}_2$ of the same type $\langle R, F, \sigma \rangle$ with a common universe M, i.e.

$$M_1 = M_2 = M,$$

and with a common fuzzy equivalence $\approx^{\mathbf{M}}$, i.e.

$$\approx^{\mathbf{M}_1} = \approx^{\mathbf{M}_2} = \approx^{\mathbf{M}} .$$

We start with a result which we use to answer the question from the title of our paper.

THEOREM 1 (SIMILARITY AND FORMULAS). For **L**-structures $\mathbf{M}_1$ and $\mathbf{M}_2$ of type $\langle R, F, \sigma \rangle$ with a common universe M and a common fuzzy equivalence $\approx^{\mathbf{M}}$, a formula φ, and valuations v_1 and v_2, we have

$$\bigotimes_{x \in \mathrm{free}(\varphi)} (v_1(x) \approx^{\mathbf{M}} v_2(x))^{|\varphi|_x} \otimes \bigotimes_{r \in R} (r^{\mathbf{M}_1} \approx r^{\mathbf{M}_2})^{|\varphi|_r} \otimes \bigotimes_{f \in F} (f^{\mathbf{M}_1} \approx f^{\mathbf{M}_2})^{|\varphi|_f}$$

$$\leq \|\varphi\|_{\mathbf{M}_1, v_1} \leftrightarrow \|\varphi\|_{\mathbf{M}_2, v_2}. \tag{13}$$

Let us comment on the technical content of Theorem 1. Note that $(r^{\mathbf{M}_1} \approx r^{\mathbf{M}_2})$ and $(f^{\mathbf{M}_1} \approx f^{\mathbf{M}_2})$ are defined by (11) and (12). Equation (13) provides an estimation: the right-hand side of (13) is at least as high as the left-hand side of (13). The right-hand side of (13) is a degree of logical equivalence of truth degrees of formula φ in two different conditions. The first condition is given by structure $\mathbf{M}_1$ and evaluation v_1, the second condition is given by structure $\mathbf{M}_2$ and evaluation v_2. The left-hand side of (13) is, basically, a degree to which these two conditions are similar. In more detail, the left-hand side of (13) is a conjunction of three truth degrees. The first one, $\bigotimes_{x \in \mathrm{free}(\varphi)} (v_1(x) \approx^{\mathbf{M}} v_2(x))^{|\varphi|_x}$, is a degree to which the elements assigned to free variables of φ by v_1 are similar to the elements assigned to those variables by v_2, modified by $|\varphi|_x$ which can be interpreted as

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a factor measuring sensitivity of φ to values of x. The second one, $\otimes_{r\in R}(r^{\mathbf{M}_1} \approx r^{\mathbf{M}_2})^{|\varphi|_r}$, is a degree to which the fuzzy relations assigned to relation symbols r from φ in $\mathbf{M}_1$ are similar to the corresponding fuzzy relations in $\mathbf{M}_2$, modified by $|\varphi|_r$ which can be interpreted as a factor measuring sensitivity of φ to changes in fuzzy relations assigned to r's. The third one, $\otimes_{f\in F}(f^{\mathbf{M}_1} \approx f^{\mathbf{M}_2})^{|\varphi|_f}$, is a degree to which the functions assigned to function symbols f from φ in $\mathbf{M}_1$ are similar to the corresponding functions in $\mathbf{M}_2$, modified by $|\varphi|_f$ which again can be interpreted as a factor measuring sensitivity of φ to changes in functions assigned to f's.

Remark 1.

(1) Note that our notion of a degree $|\varphi|_s$ of a symbol s in formula φ is related to the notion of a syntactic degree of a formula introduced by Hájek (1998). Hájek (1998) uses the syntactic degree of a formula to obtain an estimation which is a particular case of (13) and which is less tight compared to (13).

(2) Note also that the first author who realized the role of positive integers as exponents in expressions like (13) was Pavelka (1979). Namely, Pavelka introduced a concept of a logically fitting operation c in L as an operation for which there exist non-negative integers $k_1, \ldots, k_n$ such that

$$(a_1 \leftrightarrow b_1)^{k_1} \otimes \cdots \otimes (a_n \leftrightarrow b_n)^{k_n} \leq c(a_1, \ldots, a_n) \leftrightarrow c(b_1, \ldots, b_n)$$

for every $a_1, b_1, \ldots, a_n, b_n \in L$.

Example 1. Let us now present a very simple example demonstrating how Theorem 1 can be used to yield answer to our question regarding shapes of fuzzy sets. Suppose input fuzzy sets A_1 and B_1 describe customers' satisfaction with two products in such a way that for a customer $m \in M$, $A_1(m)$ and $B_1(m)$ represent degrees to which customer m is satisfied with the first and the second product, respectively. The output fuzzy set C_1 shall describe degrees of customers' satisfaction with both of the products. An appropriate choice of C_1 might be $C_1 = A_1 \otimes B_1$, with an appropriate fuzzy conjunction $\otimes$. Suppose A_2 and B_2 are different fuzzy sets describing again customers' satisfaction with the products. Then we might again consider the corresponding $C_2 = A_2 \otimes B_2$. Theorem 1 can be applied as follows. The input mapping F is represented by formula $r_A(x) \otimes r_B(x)$ where both r_A and r_B are unary relation symbols. We consider two $\mathbf{L}$-structures, $\mathbf{M}_1$ and $\mathbf{M}_2$, defined as follows. The unary fuzzy relations, i.e. fuzzy sets, $r_A^{\mathbf{M}_1}$ and $r_B^{\mathbf{M}_1}$ corresponding to r_A and r_B in $\mathbf{M}_1$ are A_1 and B_1, respectively. Likewise, the fuzzy sets $r_A^{\mathbf{M}_2}$ and $r_B^{\mathbf{M}_2}$ corresponding to r_A and r_B in $\mathbf{M}_2$ are A_2 and B_2, respectively. Notice that the formula $r_A(x) \otimes r_B(x)$ has one free variable x and that this formula describes F. Indeed, if we take valuations v_1 and v_2 such that $v_1(x) = v_2(x) = m$ then

$$||r_A(x) \otimes r_B(x)||_{\mathbf{M}_1, v_1} = (A_1 \otimes B_1)(m) = C_1(m),$$

and

$$||r_A(x) \otimes r_B(x)||_{\mathbf{M}_2, v_2} = (A_2 \otimes B_2)(m) = C_2(m).$$

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Observe that $\bigotimes_{x\in\mathrm{free}(\varphi)}(v_1(x)\approx^{\mathbf{M}} v_2(x))^{|\varphi|_x}=1$, $|r_A(x)\otimes r_B(x)|_{r_A}=1$, and $|r_A(x)\otimes r_B(x)|_{r_B}=1$. Moreover, $r_A(x)\otimes r_B(x)$ does not contain any function symbols. Therefore, (13) becomes

$$(A_1\approx A_2)\otimes(B_1\approx B_2)\leq((A_1\otimes B_1)\approx(A_2\otimes B_2)),$$

which says "if A_1 is similar to A_2 and B_1 is similar to B_2 then $A_1\otimes B_1$ is similar to $A_2\otimes B_2$". The numerical interpretation of this estimation depends on the fuzzy conjunction $\otimes$. The estimation says that small changes in shapes of the input fuzzy sets lead to small changes in shapes of the output fuzzy set and, therefore, in this case, the exact shapes of fuzzy sets do not matter.

For the sake of completeness, we present a related result which is contained in the following theorem. This time, however, the estimated truth degree is $\|t\|_{\mathbf{M}_1,v_1}\approx^{\mathbf{M}}\|t\|_{\mathbf{M}_2,v_2}$, i.e. a similarity of the element $\|t\|_{\mathbf{M}_1,v_1}$ to which a term t is evaluated in $\mathbf{M}_1$ using v_1 to the element $\|t\|_{\mathbf{M}_2,v_2}$ to which a term t is evaluated in $\mathbf{M}_2$ using v_2.

THEOREM 2 (SIMILARITY AND TERMS). For $\mathbf{L}$-structures $\mathbf{M}_1,\mathbf{M}_2$ of type $\langle R,F,\sigma\rangle$ with a common universe M and a common fuzzy equivalence $\approx^{\mathbf{M}}$, a term t, and valuations v_1 and v_2, we have

$$\bigotimes_{x\in\mathrm{var}(t)}(v_1(x)\approx^{\mathbf{M}} v_2(x))^{|t|_x}\otimes\bigotimes_{f\in F}(f^{\mathbf{M}_1}\approx f^{\mathbf{M}_2})^{|t|_f}\leq\|t\|_{\mathbf{M}_1,v_1}\approx^{\mathbf{M}}\|t\|_{\mathbf{M}_2,v_2}. \tag{14}$$

Remark 2. All the results we presented so far can be easily extended to a many-sorted case (Hájek 1998). We will freely use this fact in the following section. Recall that in a many-sorted case, we work with a nonempty set S whose elements s are called sorts. A many-sorted language $\mathcal{J}$ given by a set S of sorts (also S-sorted language) is given by the following: a nonempty set R of relation symbols, each $r\in R$ with a (possibly empty) string $s_1s_2\ldots s_n$ of sorts $s_i\in S$, $s_1\ldots s_n$ is called the arity of r; a (possibly empty) set F of function symbols each $f\in F$ with a nonempty string $s_1\ldots s_ns_{n+1}$ called arity of f; object variables, each variable x given with its sort $s\in S$, for each sort s there is a denumerable number of variables of sort s; logical connectives, truth constants and quantifiers, and auxiliary symbols as for the ordinary one-sorted language; the language is called a language with equality if R contains a relation symbol $\approx_s$ of sort s for each $s\in S$. The meaning of arities: if the arity of $r\in R$ is $s_1\ldots s_n$, then r is supposed to denote an n-ary fuzzy relation between sets of sorts $s_1,\ldots,s_n$, respectively; similarly for the arity of $f\in F$. We denote the arity of $r\in R$ by $\sigma(r)$, the arity of $f\in F$ by $\sigma(f)$, and the sort of a variable x by $\sigma(x)$. Terms are defined so as to respect the sorts of arities of function symbols. Syntactic notions like the degrees $|t|_x$, $|\varphi|_r$, etc. are defined analogously to the ordinary case. An $\mathbf{L}$-structure for an S-sorted language of type $\langle R,F,\sigma\rangle$ is a triple $\mathbf{M}=\langle M,R^{\mathbf{M}},F^{\mathbf{M}}\rangle$ where: $M=\{M_s|s\in S\}$ is a system of nonempty sets M_s (M_s is called a universe of sort s); $R^{\mathbf{M}}=\{r^{\mathbf{M}}:M_{s_1}\times\cdots\times M_{s_n}\to L|r\in R,\sigma(r)=s_1\ldots s_n\}$ of $\mathbf{L}$-relations, and a set $F^{\mathbf{M}}=\{f^{\mathbf{M}}:M_{s_1}\times\cdots\times M_{s_n}\to M_{s_{n+1}}|f\in F,\sigma(r)=s_1\ldots s_{n+1}\}$ of functions such that each $\approx^{\mathbf{M}}_s$ is an $\mathbf{L}$-equivalence relation on M_s, each $r^{\mathbf{M}}\in R^{\mathbf{M}}$ with $\sigma(r)=s_1\ldots s_n$ is compatible with $\approx^{\mathbf{K}}_{s_1},\ldots,\approx^{\mathbf{K}}_{s_n}$, and each $f^{\mathbf{M}}\in F^{\mathbf{M}}$ with $\sigma(f)=s_1\ldots s_ns_{n+1}$ is compatible with $\approx^{\mathbf{K}}_{s_1},\ldots,\approx^{\mathbf{K}}_{s_n}$, $\approx^{\mathbf{K}}_{s_{n+1}}$. In other words, an $\mathbf{L}$-structure $\mathbf{M}$ for a many-sorted language is a system of $\mathbf{L}$-relations and functions taking as arguments elements of the corresponding sorts. Given an $\mathbf{L}$-structure $\mathbf{M}$ for an S-sorted language, a valuation is a mapping assigning to any variable x an element $v(x)\in$

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$M_{\sigma(x)}$ (i.e. an element of sort s to a variable of sort s). Values of terms and formulas are then defined obviously.

4. Do, then, exact shapes of fuzzy sets matter?

In this section we discuss our question of whether exact shapes of fuzzy sets matter in five particular cases. Namely, in case of Zadeh's extension principle, composition of fuzzy relations, compositional rule of inference, fuzzy automata, and properties of fuzzy relations. In each of these cases, we show how Theorem 1 can be applied to yield an answer to our question.

4.1 Extension principle

The extension principle was formulated by Zadeh (1975) for the following purpose. Let $g : X \to Y$ be an (ordinary) function from a set X to a set Y. It can happen that instead of exact values from X, we know only their approximate linguistic descriptions. For instance, we may know that the input is "approximately x", "around 0", etc. These verbally described inputs in X can be naturally modeled by fuzzy sets in X. Now, if we still want to apply g to these approximate inputs we need a principle for extending g (which assigns elements from Y to elements from X) to $\bar{g}$ which assigns fuzzy sets in Y to fuzzy sets in X in an appropriate way. The basic requirement is that if applied to points (singletons), $\bar{g}$ and g coincide ($\bar{g}(\{1/x\}) = \{1/g(x)\}$). According to Zadeh's extension principle, $\bar{g} : L^X \to L^Y$ is defined by

$$(\bar{g}(A))(y) = \sup\{A(x) | g(x) = y\}$$

for $A \in L^X$ and $y \in Y$ (this is a generalization of so-called interval methods). Since $A \in L^X$ are basically thought of as approximations of elements of X, the question of to what extent the exact "shape of A" matters is particularly appealing. Intuitively, one expects that if the extension principle is well-proposed then $\bar{g}(A_1)$ is similar to $\bar{g}(A_2)$ whenever A_1 is similar to A_2, i.e. similar inputs are mapped to similar outputs. To see this, we can apply Theorem 1 as follows.

Let $S = \{\mathcal{X}, \mathcal{Y}\}$ and consider an S-sorted language $\mathcal{J}_{EP}$ with a unary relation symbol r_A, binary relation symbols $\approx_{\mathcal{X}}$ and $\approx_{\mathcal{Y}}$, and a unary function symbol f such that $\sigma(r_A) = \mathcal{X}$, $\sigma(\approx_{\mathcal{X}}) = \mathcal{X}\mathcal{X}$, $\sigma(\approx_{\mathcal{Y}}) = \mathcal{Y}\mathcal{Y}$, $\sigma(f) = \mathcal{X}\mathcal{Y}$. An $\mathbf{L}$-structure $\mathbf{M}$ for $\mathcal{J}_{\rm EP}$ thus consists of a set $M_{\mathcal{X}}$, a set $M_{\mathcal{Y}}$, fuzzy equivalence relations $\approx_{\mathcal{X}}^{\mathbf{M}}$ on $M_{\mathcal{X}}$ and $\approx_{\mathcal{Y}}^{\mathbf{M}}$ on $M_{\mathcal{Y}}$, and a function $f^{\mathbf{M}} : M_{\mathcal{X}} \to M_{\mathcal{Y}}$ which is compatible with $\approx_{\mathcal{X}}^{\mathbf{M}}$ and $\approx_{\mathcal{Y}}^{\mathbf{M}}$. Let EP($f$) be the formula

$$(\exists\xi)(r_A(\xi) \otimes (f(\xi) \approx_{\mathcal{Y}} \nu))$$

where ξ and ν are variables of sort $\mathcal{X}$ and $\mathcal{Y}$, respectively. Now, given $g : X \to Y$, formula EP(f) induces a mapping $G : L^X \to L^Y$ as follows. For an input fuzzy set $A \in L^X$, consider an $\mathbf{L}$-structure $\mathbf{M}$ for $J_{\rm EP}$ such that $M = \{M_{\mathcal{X}}, M_{\mathcal{Y}}\}$, $M_{\mathcal{X}} = X$, $M_{\mathcal{Y}} = Y$, $f^{\mathbf{M}} = g$, $\approx_{\mathcal{Y}}^{\mathbf{M}}$ is the identity on F (i.e. $y_1 \approx_{\mathcal{Y}}^{\mathbf{M}} y_2 = 1$ iff $y_1 = y_2$ and $y_1 \approx_{\mathcal{Y}}^{\mathbf{M}} y_2 = 0$ otherwise), and $r_A^{\mathbf{M}} = A$. Then, for any $y \in Y$, consider a valuation v such that $v(\nu) = y$ and define $G(A) \in L^Y$ by

$$(G(A))(y) = \|{\rm EP}(f)\|_{\mathbf{M},v}.$$

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It is now easy to see that $G = \bar{g}$. Indeed,

$$(G(A))(y) = \|EP(f)\|_{\mathbf{M},v} = \bigvee_{v'=_\xi v} r_A^{\mathbf{M}}(v'(\xi)) \otimes (f^{\mathbf{M}}(v'(\xi)) \approx_{\mathcal{Y}}^{\mathbf{M}} v'(\nu))$$

$$= \bigvee \{A(x) \otimes (g(x) \approx_{\mathcal{Y}}^{\mathbf{M}} y) | x \in X\}$$

$$= \bigvee \{A(x) | g(x) = y\} = (\bar{g}(A))(y).$$

That is, the function $\bar{g} : L^X \to L^Y$, which is our function F from figure 1, can be defined by formula EP(f), which is our formula φ from figure 1.

Let us now consider the question of whether the exact shapes of the input fuzzy sets $A \in L^X$ matter. If $\mathbf{M}_1$ and $\mathbf{M}_2$ are two $\mathbf{L}$-structures for $\mathcal{J}_{\mathrm{EP}}$ with a common universe and common $\mathbf{L}$-equivalence relations, then since $|\mathrm{EP}(f)|_{r_A} = 1$ and $|EP(f)|_f = 1$, (many-sorted version of) Theorem 1 yields that

$$(r_A^{\mathbf{M}_1} \approx r_A^{\mathbf{M}_2}) \otimes (f^{\mathbf{M}_1} \approx f^{\mathbf{M}_2}) \le \|\mathrm{EP}(f)\|_{\mathbf{M}_1,v} \leftrightarrow \|\mathrm{EP}(f)\|_{\mathbf{M}_2,v}. \tag{15}$$

In particular, if we take $\mathbf{M}_1$ and $\mathbf{M}_2$ such that $M_1 = M_2 = \{M_{\mathcal{X}}, M_{\mathcal{Y}}\}$, $M_{\mathcal{X}} = X$, $M_{\mathcal{Y}} = Y$, $f^{\mathbf{M}_1} = f^{\mathbf{M}_2} = g$, $\approx_{\mathcal{Y}}^{\mathbf{M}_1}$ and $\approx_{\mathcal{Y}}^{\mathbf{M}_2}$ are identities on Y and $r_A^{\mathbf{M}_1} = A_1 \in L^X$ and $r_A^{\mathbf{M}_2} = A_2 \in L^X$, (15) becomes $(A_1 \approx A_2) \le (\bar{g}(A_1))(y) \leftrightarrow (\bar{g}(A_2))(y)$, for each $y \in Y$, whence we get

THEOREM 3. For a function $g : X \to Y$ and $A_1, A_2 \in L^X$ we have

$$(A_1 \approx A_2) \le (\bar{g}(A_1) \approx \bar{g}(A_2)).$$

Proof. Directly from the above considerations by recalling that $(\bar{g}(A_1) \approx \bar{g}(A_2)) = \bigwedge_{y \in Y} (\bar{g}(A_1))(y) \leftrightarrow (\bar{g}(A_2))(y)$. □

That is, similar input fuzzy sets A_1 and A_2 are mapped to similar output fuzzy sets $\bar{g}(A_1)$ and $\bar{g}(A_2)$. Therefore, we have

Answer (extension principle): For functions defined by extension principle, the exact shapes of fuzzy sets representing approximate inputs do not matter.

4.2 Fuzzy relational products

We now consider relational products of fuzzy relations (called also compositions of fuzzy relations). Products of fuzzy relations have been extensively discussed by Bandler and Kohout, see e.g. (Kohout and Kim 2002) for a survey. Suppose we have sets X, Y and Z, equipped with $\mathbf{L}$-equivalences $\approx^X$, $\approx^Y$, and $\approx^Z$, respectively. Furthermore, we have $\mathbf{L}$-relations R between X and Y and S between F and Y compatible with the $\mathbf{L}$-equivalences. For R and S, we introduce new $\mathbf{L}$-relations between X and Z denoted by $R \circ S$, $R \lhd S$, $R \rhd S$, and $R \Box S$ which are called the $\circ$-composition, $\lhd$-composition, $\rhd$-composition, and

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$\Box$-composition, respectively. The **L**-relations are defined by

$$(R \circ S)(x,z) = \bigvee_{y \in Y} (R(x,y) \otimes S(y,z)),$$

$$(R \lhd S)(x,z) = \bigwedge_{y \in Y} (R(x,y) \to S(y,z)),$$

$$(R \rhd S)(x,z) = \bigwedge_{y \in Y} (S(y,z) \to R(x,y)),$$

$$(R \Box S)(x,z) = \bigvee_{y \in Y} (R(x,y) \leftrightarrow S(y,z)).$$

As an illustrative example, let X be a set of patients, Y be a set of symptoms (of diseases), and Z be a set of diseases. Let R be a fuzzy relation between X and Y, S be a fuzzy relation between Y and Z. R may represent results of a medical examination, i.e. for a patient $x \in X$ and a symptom $y \in Y$ (e.g. a headache), $R(x,y)$ is the truth degree to which x has headache (notice that typically, R is a non-crisp fuzzy relation). Similarly, S may represent expert knowledge (which can be found in medical literature), i.e. for a symptom $y \in Y$ and a disease $z \in Z$, $S(y,z)$ is the truth degree to which y is a symptom of z (again, S is a typical fuzzy relation). Then, for instance, $(R \Box S)(x,z)$ is the truth degree of "for each symptom $y \in Y$, patient x has y iff y is a symptom of z". That is, $(R \Box S)(x,z)$ may be understood as the degree to which patient x has disease z. Since both R and S are obtained by subjective judgment, it is important to know to what extent it matters if, instead of R and S, we are supplied with other fuzzy relations R' and S' which are similar to R and S, respectively.

In order to apply Theorem 1, we can proceed as follows. Let $\mathcal{J}_{\text{comp}}$ be a many-sorted language with sorts $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$, of type $\langle R, F, \sigma \rangle$ with binary $r, s \in R$ such that $\sigma(r) = \mathcal{X}\mathcal{Y}$ and $\sigma(s) = \mathcal{Y}\mathcal{Z}$, let ξ, ν, ς be variables of sorts $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$, respectively. Then, we take a formula $\varphi_*(\xi, \varsigma)$ which represents $*$-composition of fuzzy relations as follows (see later for particular formulas $\varphi_\circ(\xi, \varsigma)$, $\varphi_\lhd(\xi, \varsigma)$, $\varphi_\rhd(\xi, \varsigma)$, $\varphi_\Box(\xi, \varsigma)$): for an **L**-relation R between X and Y and an **L**-relation S between Y and Z, consider an **L**-structure **M** given by $r^{\mathbf{M}} = R$, $s^{\mathbf{M}} = S$, $\approx^{\mathbf{M}}_{\mathcal{X}}$ be $\approx^X$, $\approx^{\mathbf{M}}_{\mathcal{Y}}$ be $\approx^Y$, and $\approx^{\mathbf{M}}_{\mathcal{Z}}$ be $\approx^Z$. φ_* induces a binary **L**-relation $R*S$ between X and Z defined by

$$(R*S)(x,z) = \|\varphi\|_{\mathbf{M},v} \tag{16}$$

where $x \in X$, $z \in Z$, and v is a valuation such that $v(\xi) = x$, $v(\varsigma) = z$. In order to represent the above-defined products, we use the following formulas:

for $\circ$-composition:

$$\varphi_\circ \ \text{ is } \ (\exists \nu)(r(\xi, \nu) \otimes s(\nu, \varsigma));$$

for $\lhd$-composition:

$$\varphi_\lhd \ \text{ is } (\forall \nu)(r(\xi, \nu) \Rightarrow s(\nu, \varsigma));$$

for $\rhd$-composition:

$$\varphi_\rhd \ \text{ is } (\forall \nu)(s(\nu, \varsigma) \Rightarrow r(\xi, \nu));$$

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for □-composition:

$$\varphi_{\Box} \text{ is } (\forall \nu)(r(\xi, \nu) \Leftrightarrow s(\nu, \varsigma)).$$

As one can immediately see, the **L**-relations defined from $\varphi_{\circ}$, $\varphi_{\lhd}$, $\varphi_{\rhd}$, and $\varphi_{\Box}$ by (16) are just the above-defined products $R\circ S$, $R\lhd S$, $R \rhd S$, and $R\Box S$, respectively. Then, since $|\varphi_{*}|_r = 1$ and $|\varphi_{*}|_s = 1$ for each product type $*$, Theorem 1 becomes

THEOREM 4

$$(R_1 \approx R_2)\otimes(S_1 \approx S_2) \leq (R_1\circ S_1) \approx (R_2\circ S_2),$$

$$(R_1 \approx R_2)\otimes(S_1 \approx S_2) \leq (R_1\lhd S_1) \approx (R_2\lhd S_2),$$

$$(R_1 \approx R_2)\otimes(S_1 \approx S_2) \leq (R_1 \rhd S_1) \approx (R_2 \rhd S_2),$$

$$(R_1 \approx R_2)\otimes(S_1 \approx S_2) \leq (R_1\Box S_1) \approx (R_2\Box S_2).$$

In words, similar input fuzzy relations lead to similar relational products. Therefore, we have again

Answer (**fuzzy relational products**)**:** For fuzzy relational products, the exact shapes of input fuzzy relations do not matter.

4.3 Compositional rule of inference

Recall that a compositional rule of inference allows us to infer a fuzzy set CRI(A, R) in a universe Y from a given fuzzy set A in a universe X and a fuzzy relation R between X and Y by

$$[\mathrm{CRI}(A, R)](y) = \bigvee_{x\in X} A(x) \otimes R(x, y)$$

for each $y \in Y$.

Remark 3. Note that R usually represents a vaguely described relationship between elements from X and elements from Y. R often results from well-known if-then rules like

$$\text{if } x \text{ is } A_i \text{ then } y \text{ is } B_i,$$

where $A_i \in L^X$, $B_i \in L^Y$, $i = 1, \dots, n$. Note also that CRI is the rule of inference used in Mamdani-type fuzzy controllers.

Since it is a well-known fact that the compositional rule of inference is a particular case of $\circ$-product of fuzzy relations, we omit further details in this case. We just note that our formula φ in this case is

$$(\exists\xi)(r_A(\xi) \otimes r_R(\xi, \nu)).$$

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Theorem 1 then yields the following estimation:

THEOREM 5. For compositional rule of inference, we have

$$(A_1 \approx A_2) \otimes (R_1 \approx R_2) \leq (\mathrm{CRI}(A_1, R_1) \approx \mathrm{CRI}(A_2, R_2)).$$

This gives, again,

Answer (compositional rule of inference): For compositional rule of inference, the exact shapes of input fuzzy set A and input fuzzy relation R do not matter.

4.4 Fuzzy automata

The notion of a fuzzy automaton generalizes that of a non-deterministic automaton. The main aim is to have a device for recognizing fuzzy languages, i.e. languages to which words belong to truth degrees not necessarily equal to 0 or 1. Such languages are quite natural: a language consisting of words that are being used only seldom, a language consisting of words that are difficult to pronounce, a language consisting of sequences of notes of music which represent a nice sound, etc.

Recall that an ordinary non-deterministic automaton over an alphabet Σ (a set of elementary symbols) is given by a finite set Q of states, subsets $Q_I \subseteq Q$ of initial states and $Q_F \subset Q$ of final states, and a relation $\delta \subseteq Q \times \Sigma \times Q$. δ is called a transition relation; the meaning is the following: if the current state of the automaton is $q \in Q$ and the current symbol on input is $s \in \Sigma$, then the automaton can proceed so that its next state will be any $q' \in Q$ such that $\langle q, s, q' \rangle \in \delta$. An input word $s_1 \ldots s_n$ ($s_i \in \Sigma$) is accepted by the automaton iff there is a sequence $q_1, \ldots, q_{n+1}$ of states such that q_1 is an initial state, q_{n+1} is a final state, and $\langle q_i, s_1, q_{i+1} \rangle \in \delta$ (transition). The set of all words accepted by the automaton is called the language recognized by the automaton.

In a fuzzy setting, we have the following. An $\mathbf{L}$-automaton $\mathcal{M}$ over a finite alphabet Σ is given by a finite set Q of states, an $\mathbf{L}$-set Q_I in Q (for $q \in Q$, $Q_I(q)$ is the degree to which q is an initial state); an $\mathbf{L}$-set Q_F in Q (for $q \in Q$, $Q_F(q)$ is the degree to which q is a final state); an $\mathbf{L}$-relation δ between Q, Σ, and Q (for $q, q' \in Q$ and $s \in \Sigma$, $\delta(q, s, q')$ is the degree to which the $\mathbf{L}$-automaton can transfer from q to q' if the current input symbol is s). Then, for an input word $s_1 \ldots s_n$ we define the degree $(\mathcal{L}(\mathcal{M}))(s_1 \ldots s_n)$ to which $\mathcal{M}$ accepts $s_1 \ldots s_n$ by

$$(\mathcal{L}(\mathcal{M}))(s_1 \ldots s_n) = \bigvee_{q_1, \ldots, q_{n+1} \in Q} Q_I(q_1) \wedge \delta(q_1, s_1, q_2) \wedge \cdots \wedge \delta(q_n, s_n, q_{n+1}) \wedge Q_F(q_{n+1}).$$

The fuzzy set $\mathcal{L}(\mathcal{M})$ is called an $\mathbf{L}$-language recognized by $\mathcal{M}$. We can easily see that for $\mathbf{L} = 2$ we get the ordinary notion of a non-deterministic automaton and the corresponding language.

An $\mathbf{L}$-automaton can be viewed as a many-sorted $\mathbf{L}$-structure: consider an S-sorted language $\mathcal{J}_{\mathrm{Aut}}$ where $S = \{\mathcal{Q}, \mathcal{S}\}$, $R = \{r_\delta, r_{Q_I}, r_{Q_F}\}$, $F = \emptyset$. Then an $\mathbf{L}$-automaton $\mathcal{M}$ can be viewed as an $\mathbf{L}$-structure $\mathbf{M}$ for $\mathcal{J}_{\mathrm{Aut}}$ where $r_\delta^{\mathbf{M}} = \delta$, $r_{Q_I}^{\mathbf{M}} = Q_I$, and $r_{Q_F}^{\mathbf{M}} = Q_F$. Using the rules for evaluating truth degrees of formulas we easily see that the degree $(\mathcal{L}(\mathcal{M}))(s_1 \ldots s_n)$ to which $\mathcal{M}$ accepts $s_1 \ldots s_n$ equals the truth degree of a formula

$$(\exists \xi_1, \ldots, \xi_{n+1})[r_{Q_I}(\xi_1) \mathbin{\&} r_\delta(\xi_1, \nu_1, \xi_2) \mathbin{\&} \cdots \mathbin{\&} r_\delta(\xi_n, \nu_n, \xi_{n+1}) \mathbin{\&} r_{Q_F}(\xi_1)]$$

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for a valuation v such that $v(\nu_i) = s_i$. Denoting this formula accept_n we thus have

$$(\mathcal{L}(\mathcal{M}))(s_1 \ldots s_n) = \|\text{accept}_n\|_{\mathbf{M},v}.$$

Formula accept_n is our formula φ. This setting enables us to apply (many-sorted version of) Theorem 1 and we thus get

THEOREM 6. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be $\mathbf{L}$-automata over an alphabet Σ; then we have

$$(Q_{I1} \approx Q_{I2}) \otimes (\delta_1 \approx \delta_2) \otimes (Q_{F1} \approx Q_{F2}) \leq (\mathcal{L}(\mathcal{M}_1) \approx \mathcal{L}(\mathcal{M}_2)).$$

Proof. Let $\mathbf{M}_1$ and $\mathbf{M}_2$ be the $\mathbf{L}$-structures for $\mathcal{J}_{\text{Aut}}$ corresponding to $\mathcal{M}_1$ and $\mathcal{M}_2$ (i.e. $r_{Q_I}^{\mathbf{M}_i} = Q_{Ii}$, $r_{\delta}^{\mathbf{M}_i} = \delta_i$, and $r_{Q_F}^{\mathbf{M}_i} = Q_{Fi}$, $i = 1, 2$). Since $|\text{accept}_n|_{r_{Q_I}} = 1$, $|\text{accept}_n|_{r_\delta} = 1$, and $|\text{accept}_n|_{r_{Q_F}} = 1$, (many-sorted version of) Theorem 1 yields

$$(r_{Q_I}^{\mathbf{M}_1} \approx r_{Q_I}^{\mathbf{M}_2}) \otimes (r_{\delta}^{\mathbf{M}_1} \approx r_{\delta}^{\mathbf{M}_2}) \otimes (r_{Q_F}^{\mathbf{M}_1} \approx r_{Q_F}^{\mathbf{M}_2}) \leq \|\text{accept}_n\|_{\mathbf{M}_1,v} \leftrightarrow \|\text{accept}_n\|_{\mathbf{M}_2,v}.$$

Since for v such that $v(\nu_i) = s_i$ $(i = 1, \ldots, n)$ we have $\|\text{accept}_n\|_{\mathbf{M}_i,v} = (\mathcal{L}(\mathcal{M}_i))(s_1 \ldots s_n)$, recalling that

$$(\mathcal{L}(\mathcal{M}_1) \approx \mathcal{L}(\mathcal{M}_2)) = \bigwedge_{n \in \mathbf{N}, s_1, \ldots, s_n \in \Sigma} (\mathcal{L}(\mathcal{M}_1))(s_1 \ldots s_n) \leftrightarrow (\mathcal{L}(\mathcal{M}_2))(s_1 \ldots s_n),$$

the assertion follows. □

That is, two $\mathbf{L}$-automata with pairwise similar fuzzy sets of input states, output states, and similar transition relation recognize similar languages. This gives

Answer (fuzzy automata): For fuzzy automata, the exact membership degrees of input and output states and exact membership degrees of transition relation do not matter.

4.5 Properties of fuzzy relations

Properties of binary relations in a set, such as reflexivity, symmetry, etc. are our next topic. We are going to consider graded properties which is truly in the spirit of fuzzy approach, see e.g. (Gottwald 2001, Belohlavek 2002). We suppose we are given a binary $\mathbf{L}$-relation R in a set X i.e. $R : X \times X \to L$. For the purpose of our illustration, we limit our consideration to degrees $\text{Ref}(R)$ of reflexivity, $\text{Sym}(R)$ of symmetry, $\text{Tra}(R)$ of transitivity of R. These degrees are defined by

$$\text{Ref}(R) = \bigwedge_{x \in X} R(x, x),$$
$$\text{Sym}(R) = \bigwedge_{x,y \in X} (R(x, y) \to R(y, x)),$$
$$\text{Tra}(R) = \bigwedge_{x,y,z \in X} (R(x, y) \otimes R(y, z)) \to R(x, z),$$

respectively. We say that R is reflexive, symmetric, transitive, if $\text{Ref}(R) = 1$, $\text{Sym}(R) = 1$, $\text{Tra}(R) = 1$, respectively, which means (easy observation) that for each $x, y, z \in X$,

$$R(x, x) = 1, \quad R(x, y) = R(y, x), \quad R(x, y) \otimes R(y, z) \leq R(x, z),$$

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respectively. In this setting, our question of whether the exact shapes of fuzzy sets matter becomes whether the exact membership degrees of R matter for R to have a given property. That is, is it true that similar fuzzy relations have similar properties?

In general, the notion of a property of a binary fuzzy relation can be approached as follows. Consider a language $\mathcal{J}_{\text{bin}}$ of type $\langle R, F, \sigma\rangle$ with binary $r \in R$. Let φ be a formula that contains no relation or function symbols except for r, $\approx$, and has no free variables. The property of binary fuzzy relations which corresponds to φ is defined as follows: let R be an $\mathbf{L}$-relation in X and consider an $\mathbf{L}$-structure $\mathbf{M}$ given by $r^{\mathbf{M}} = R$, and $\approx^{\mathbf{M}}$ being identity in X. Call $\mathbf{M}$ the $\mathbf{L}$-structure corresponding to R. Then, a degree $\varphi(R)$ to which the fuzzy relation R satisfies property corresponding to φ can be defined by

$$\varphi(R) = \|\varphi\|_{\mathbf{M}}.$$

Note that since φ has no free variables, the degree $\|\varphi\|_{\mathbf{M}}$ does not depend on a particular valuation v. Then, as a direct consequence of Theorem 1, we get

Lemma 2. For φ we have

$$(R_1 \approx R_2)^{|\varphi|_r} \otimes \varphi(R_1) \leq \varphi(R_2).$$

Consider now the following particular formulas of $\mathcal{J}_{\text{bin}}$:

$$\begin{aligned}
&\text{Ref is } (\forall\xi)(r(\xi,\xi)),\\
&\text{Sym is } (\forall\xi,\nu)(r(\xi,\nu) \Rightarrow r(\nu,\xi)),\\
&\text{Tra is } (\forall\xi,\nu,\varsigma)((r(\xi,\nu) \otimes r(\nu,\xi)) \Rightarrow r(\xi,\varsigma)).
\end{aligned}$$

A direct verification shows that

$$\|\text{Ref}\|_{\mathbf{M}} = \text{Ref}(R), \quad \|\text{Sym}\|_{\mathbf{M}} = \text{Sym}(R), \quad \|\text{Tra}\|_{\mathbf{M}} = \text{Tra}(R).$$

That is, the above introduced degrees to which R is reflexive, symmetric, and transitive, are the truth degrees of formulas Ref, Sym and Tra. Therefore, the discussed properties can be described by logical formulas. As a consequence of Lemma 2, we get

Theorem 7. For $\mathbf{L}$-relations R_1, R_2 in X we have

$$\begin{aligned}
(R_1 \approx R_2) \otimes \text{Ref}(R_1) &\leq \text{Ref}(R_2),\\
(R_1 \approx R_2)^2 \otimes \text{Sym}(R_1) &\leq \text{Sym}(R_2),\\
(R_1 \approx R_2)^3 \otimes \text{Tra}(R_1) &\leq \text{Tra}(R_2).
\end{aligned}$$

Proof. The proof follows by Lemma 2 observing that $|\text{Ref}|_r = 1$, $|\text{Sym}|_r = 2$, $|\text{Tra}|_r = 3$. □

Note that the estimations from Theorem 7 say that if R_1 and R_2 are similar and R_1 is reflexive then R_2 is reflexive, too; if R_1 and R_2 are similar and R_1 is symmetric then R_2 is symmetric, too, and the like for transitivity. In this sense, similar fuzzy relations have similar properties.

However, as we can see from Theorem 7, sensitivity to changes in the membership function of R is different for different properties. The least sensitive property is reflexivity (since the exponent in $(R_1 \approx R_2)$ is 1), the most sensitive is transitivity (since the exponent in

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$(R_1 \approx R_2)^3$ is 3). As an example, let **L** be the Łukasiewicz structure on the real unit interval $[0, 1]$. R_1 be a fuzzy relation in X such that $\mathrm{Ref}(R_1) = 0.8$ (i.e. R_1 is almost reflexive). Let R_2 be another fuzzy relation in X which is similar to R_1 in degree 0.9, i.e. $R_1 \approx R_2 = 0.9$. Theorem 7 then yields

$$0.7 = 0.9 \otimes 0.8 = (R_1 \approx R_2) \otimes \mathrm{Ref}(R_1) \leq \mathrm{Ref}(R_2),$$

i.e. the degree to which R_2 is reflexive is at least 0.7. Suppose that the property in question is transitivity instead of reflexivity. That is, we have $\mathrm{Tra}(R_1) = 0.8$ (i.e. R_1 is almost transitive) and, again, $R_1 \approx R_2 = 0.9$. Theorem 7 then yields

$$0.5 = 0.7 \otimes 0.8 = 0.9^3 \otimes 0.8 = (R_1 \approx R_2) \otimes \mathrm{Tra}(R_1) \leq \mathrm{Tra}(R_2),$$

i.e. the degree to which R_2 is transitive is guaranteed to be at least 0.5 only. Therefore, we can summarize our

Answer (properties of fuzzy relations): For properties of fuzzy relations, the exact membership degrees of fuzzy relations do not matter. However, some properties are more sensitive to changes in membership degrees of fuzzy relations than others.

5. Conclusions

This paper attempts to answer a fundamental question in fuzzy relational modeling, namely, whether exact shapes of fuzzy sets matter. We presented general theorems which enable us to answer the question provided the fuzzy sets involved are processed in way which can be described by logical formulas. While the latter condition imposes limits on the applicability of our results, the condition is satisfied by several important instances of fuzzy logic modeling, including Zadeh's extension principle, fuzzy relational products, compositional rule of inference, fuzzy automata, which we used for demonstration of our results. Our answer to the question from the title of the present paper can be summarized as follows:

Answer to the question: There is a general tool, presented in this paper, which enables us to answer the question in specific situations. As demonstrated in this paper, for several important instances of fuzzy logic modeling, exact shapes of fuzzy sets do not matter.

Acknowledgements

The author is also affiliated with Palacký University, Olomouc, Czech Republic. Supported by institutional support, research plan MSM 6198959214.

References

Belohlavek, R., 2002, *Fuzzy Relational Systems. Foundations and Principles* (New York: Kluwer).
Goguen, J.A., 1967, L-fuzzy sets. *Journal of Mathematical Analysis and Applications*, **18**, pp. 145–174.
Gottwald, S., 2001, *A Treatise on Many-Valued Logics* (Baldock, Hertfordshire, England: Research Studies Press).
Hájek, P., 1998, *Metamathematics of Fuzzy Logic* (Dordrecht: Kluwer).
Höhle, U., 1996, On the fundamentals of fuzzy set theory. *Journal of Mathematical Analysis and Applications*, **201**, pp. 786–826.
Klir, G.J. and Yuan, B., 1995, *Fuzzy Sets and Fuzzy Logic* (Upper Sadle River: Prentice Hall).

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Kohout, L.J. and Kim, E., 2002, The role of BK-products of relations in soft computing. *Soft Computing*, **6**(2), pp. 92–115.
Pavelka, J., 1979, On fuzzy logic I II, III. *Z Math Logik Grundl Math*, **25**, pp. 45–52, 119–139, 447–464.
Zadeh, L.A., 1965, Fuzzy sets. *Information and Control*, **8**(3), pp. 338–353.
Zadeh, L.A., 1975, The concept of a linguistic variable and its application to approximate reasoning I, II, III. *Information Sciences*, **8**(3), pp. 199–251 301–357, **9,** 43–80.

Appendix: Proofs

This section provides auxiliary results and the proofs of Theorem 1 and 2.

Lemma 3. For an **L**-structure **M** of type $\langle R, F, \sigma\rangle$, a term t, and **M**-valuations v_1 and v_2 we have

$$\bigotimes_{x\in \mathrm{var}(t)} (v_1(x) \approx^{\mathbf{M}} v_2(x))^{|t|_x} \leq \|t\|_{\mathbf{M},v_1} \approx^{\mathbf{M}} \|t\|_{\mathbf{M},v_2}.$$

Proof. The proof goes by induction on complexity of t:

Induction base: If t is a variable x_i then $|t|_{x_i} = 1$ and the assertion follows from the compatibility of $f^{\mathbf{M}}$.

Induction step: Let $t = f(t_1, \ldots, t_m)$ and let the assertion be valid for t_j, i.e. $\bigotimes_{x\in \mathrm{var}(t_j)} (v_1(x) \approx^{\mathbf{M}} v_2(x))^{|t_j|_x} \leq \|t_j\|_{\mathbf{M},v_1} \approx^{\mathbf{M}} \|t_j\|_{\mathbf{M},v_2}$ $(j = 1, \ldots, m)$. Then we have

$$\begin{aligned}
\bigotimes_{x\in \mathrm{var}(t)} (v_1(x) \approx^{\mathbf{M}} v_2(x))^{|t|_x} &= \bigotimes_{x\in \mathrm{var}(t)} (v_1(x) \approx^{\mathbf{M}} v_2(x))^{\sum_{j=1}^{m} |t_j|_x} \\
&= \bigotimes_{x\in \mathrm{var}(t)} (v_1(x) \approx^{\mathbf{M}} v_2(x))^{|t_1|_x} \otimes \cdots \otimes (v_1(x) \approx^{\mathbf{M}} v_2(x))^{|t_m|_x} \\
&= \bigotimes_{j=1}^{m} \bigotimes_{x\in \mathrm{var}(t)} (v_1(x) \approx^{\mathbf{M}} v_2(x))^{|t_j|_x} \\
&= \bigotimes_{j=1}^{m} \bigotimes_{x\in \mathrm{var}(t_j)} (v_1(x) \approx^{\mathbf{M}} v_2(x))^{|t_j|_x} \\
&\leq \bigotimes_{j=1}^{m} \left(\|t_j\|_{\mathbf{M},v_1} \approx^{\mathbf{M}} \|t_j\|_{\mathbf{M},v_2}\right) \\
&\leq f^{\mathbf{M}}\left(\|t_1\|_{\mathbf{M},v_1}, \ldots, \|t_1\|_{\mathbf{M},v_1}\right) \approx^{\mathbf{M}} f^{\mathbf{M}}\left(\|t_1\|_{\mathbf{M},v_2}, \ldots, \|t_1\|_{\mathbf{M},v_2}\right) \\
&= \|t\|_{\mathbf{M},v_1} \approx^{\mathbf{M}} \|t\|_{\mathbf{M},v_2},
\end{aligned}$$

by compatibility of $f^{\mathbf{M}}$. □

Lemma 3 shows how the similarity degree $\|t\|_{\mathbf{M},v_1} \approx^{\mathbf{M}} \|t\|_{\mathbf{M},v_2}$ of values of a term t under two valuations v_1 and v_2 can be estimated by similarity degrees $v_1(x) \approx^{\mathbf{M}} v_2(x)$ of values of v_1 and v_2. The next lemma shows an analogous result for formulas: it gives an estimation of $\|\varphi\|_{\mathbf{M},v_1} \leftrightarrow \|\varphi\|_{\mathbf{M},v_2}$.

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LEMMA 4. For an **L**-structure **M** of type $\langle R, F, \sigma\rangle$, a formula φ, and **M**-valuations v_1 and v_2 we have

$$\bigotimes_{x\in\mathrm{free}(\varphi)}(v_1(x)\approx^{\mathbf{M}} v_2(x))^{|\varphi|_x} \leq \|\varphi\|_{\mathbf{M},v_1} \leftrightarrow \|\varphi\|_{\mathbf{M},v_2}.$$

Proof. First, observe that the assertion is equivalent to

$$\bigotimes_{x\in\mathrm{free}(\varphi)}(v_1(x)\approx^{\mathbf{M}} v_2(x))^{|\varphi|_x}\otimes\|\varphi\|_{\mathbf{M},v_1} \leq \|\varphi\|_{\mathbf{M},v_2}.$$

We prove the assertion by induction over the complexity of φ.

The assertion is trivial if φ is a symbol of a truth degree. If $\varphi = r(t_1, \ldots, t_m)$ then, by Lemma 3, we have

$$\begin{aligned}\bigotimes_{x\in\mathrm{free}(\varphi)}(v_1(x)\approx^{\mathbf{M}} v_2(x))^{|\varphi|_x} &= \bigotimes_{x\in\mathrm{free}(\varphi)}(v_1(x)\approx^{\mathbf{M}} v_2(x))^{|t_1|_x+\cdots+|t_m|_x}\\ &= \bigotimes_{x\in\mathrm{var}(t_1)}(v_1(x)\approx^{\mathbf{M}} v_2(x))^{|t_1|_x}\otimes\cdots\otimes \bigotimes_{x\in\mathrm{var}(t_m)}(v_1(x)\approx^{\mathbf{M}} v_2(x))^{|t_m|_x}\\ &\leq \left(\|t_1\|_{\mathbf{M},v_1}\approx^{\mathbf{M}}\|t_1\|_{\mathbf{M},v_2}\right)\otimes\cdots\otimes\left(\|t_m\|_{\mathbf{M},v_1}\approx^{\mathbf{M}}\|t_m\|_{\mathbf{M},v_2}\right)\\ &\leq r^{\mathbf{M}}\left(\|t_1\|_{\mathbf{M},v_1},\ldots,\|t_m\|_{\mathbf{M},v_1}\right)\leftrightarrow r^{\mathbf{M}}\left(\|t_1\|_{\mathbf{M},v_2},\ldots,\|t_m\|_{\mathbf{M},v_2}\right)\\ &= \|\varphi\|_{\mathbf{M},v_1}\leftrightarrow\|\varphi\|_{\mathbf{M},v_2}.\end{aligned}$$

Let the assertion be valid for φ and ψ. We have

$$\begin{aligned}\bigotimes_{x\in\mathrm{free}(\varphi\wedge\psi)}(v_1(x)\approx^{\mathbf{M}} v_2(x))^{|\varphi\wedge\psi|_x} &= \bigotimes_{x\in\mathrm{free}(\varphi\wedge\psi)}(v_1(x)\approx^{\mathbf{M}} v_2(x))^{\max(|\varphi|_x,|\psi|_x)}\\ &\leq \bigotimes_{x\in\mathrm{free}(\varphi)}(v_1(x)\approx^{\mathbf{M}} v_2(x))^{|\varphi|_x}\\ &\leq \|\varphi\|_{\mathbf{M},v_1}\leftrightarrow\|\varphi\|_{\mathbf{M},v_2},\end{aligned}$$

and similarly,

$$\bigotimes_{x\in\mathrm{free}(\varphi\wedge\psi)}(v_1(x)\approx^{\mathbf{M}} v_2(x))^{|\varphi\wedge\psi|_x} \leq \|\psi\|_{\mathbf{M},v_1}\leftrightarrow\|\psi\|_{\mathbf{M},v_2}.$$

Since

$$\begin{aligned}&(\|\varphi\|_{\mathbf{M},v_1}\leftrightarrow\|\varphi\|_{\mathbf{M},v_2})\wedge(\|\psi\|_{\mathbf{M},v_1}\leftrightarrow\|\psi\|_{\mathbf{M},v_2})\\ \leq\ &(\|\varphi\|_{\mathbf{M},v_1}\wedge\|\psi\|_{\mathbf{M},v_1})\leftrightarrow(\|\varphi\|_{\mathbf{M},v_2}\wedge\|\psi\|_{\mathbf{M},v_2})\\ =\ &\|\varphi\wedge\psi\|_{\mathbf{M},v_1}\leftrightarrow\|\varphi\wedge\psi\|_{\mathbf{M},v_2},\end{aligned}$$

we have

$$\bigotimes_{x\in\mathrm{free}(\varphi\wedge\psi)}(v_1(x)\approx^{\mathbf{M}} v_2(x))^{|\varphi\wedge\psi|_x} \leq \|\varphi\wedge\psi\|_{\mathbf{M},v_1}\leftrightarrow\|\varphi\wedge\psi\|_{\mathbf{M},v_2},$$

i.e. the assertion is valid for $\varphi\wedge\phi$.

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In order to show that the assertion is valid for $\varphi \Cup \phi$, we have to show

$$\bigotimes_{x\in \mathrm{free}(\varphi\Cup\psi)} (v_1(x) \approx^{\mathbf{M}} v_2(x))^{|\varphi\Cup\psi|_x} \otimes \|\varphi \Cup \psi\|_{\mathbf{M},v_1} \leq \|\varphi \Cup \psi\|_{\mathbf{M},v_2}.$$

We have

$$\begin{aligned}
&\bigotimes_{x\in \mathrm{free}(\varphi\Cup\psi)} (v_1(x) \approx^{\mathbf{M}} v_2(x))^{|\varphi\Cup\psi|_x} \otimes \|\varphi \Cup \psi\|_{\mathbf{M},v_1} \\
&= \Big(\bigotimes_{x\in \mathrm{free}(\varphi\Cup\psi)} (v_1(x) \approx^{\mathbf{M}} v_2(x))^{\max(|\varphi|_x,|\psi|_x)} \otimes \|\varphi\|_{\mathbf{M},v_1}\Big) \\
&\quad \vee \Big(\bigotimes_{x\in \mathrm{free}(\varphi\Cup\psi)} (v_1(x) \approx^{\mathbf{M}} v_2(x))^{\max(|\varphi|_x,|\psi|_x)} \otimes \|\psi\|_{\mathbf{M},v_1}\Big) \\
&\leq \Big(\bigotimes_{x\in \mathrm{free}(\varphi)} (v_1(x) \approx^{\mathbf{M}} v_2(x))^{|\varphi|_x} \otimes \|\varphi\|_{\mathbf{M},v_1}\Big) \\
&\quad \vee \Big(\bigotimes_{x\in \mathrm{free}(\psi)} (v_1(x) \approx^{\mathbf{M}} v_2(x))^{|\psi|_x} \otimes \|\psi\|_{\mathbf{M},v_1}\Big) \\
&\leq \|\varphi\|_{\mathbf{M},v_2} \vee \|\psi\|_{\mathbf{M},v_2} = \|\varphi \Cup \psi\|_{\mathbf{M},v_2}
\end{aligned}$$

which was to be proved.

Consider $\varphi \otimes \psi$*: We have*

$$\begin{aligned}
&\Big(\bigotimes_{x\in \mathrm{free}(\varphi\otimes\psi)} (v_1(x) \approx^{\mathbf{M}} v_2(x))^{|\varphi\otimes\psi|_x}\Big) \otimes \|\varphi \otimes \psi\|_{\mathbf{M},v_1} \\
&= \Big(\bigotimes_{x\in \mathrm{free}(\varphi\otimes\psi)} (v_1(x) \approx^{\mathbf{M}} v_2(x))^{|\varphi|_x+|\psi|_x}\Big) \otimes \|\varphi\|_{\mathbf{M},v_1} \otimes \|\varphi\|_{\mathbf{M},v_1} \\
&\leq \Big(\bigotimes_{x\in \mathrm{free}(\varphi)} (v_1(x) \approx^{\mathbf{M}} v_2(x))^{|\varphi|_x} \otimes \|\varphi\|_{\mathbf{M},v_1}\Big) \\
&\quad \otimes \Big(\bigotimes_{x\in \mathrm{free}(\psi)} (v_1(x) \approx^{\mathbf{M}} v_2(x))^{|\psi|_x} \otimes \|\psi\|_{\mathbf{M},v_1}\Big) \\
&\leq \|\varphi\|_{\mathbf{M},v_2} \otimes \|\psi\|_{\mathbf{M},v_2} = \|\varphi \otimes \psi\|_{\mathbf{M},v_2},
\end{aligned}$$

i.e. the assertion holds for $\varphi \otimes \psi$.

We show that the assertion is true for $\varphi \Rightarrow \psi$. We have to show

$$\left(\bigotimes_{x\in \mathrm{free}(\varphi\Rightarrow\psi)} (v_1(x) \approx^{\mathbf{M}} v_2(x))^{|\varphi\Rightarrow\psi|_x}\right) \otimes \|\varphi \Rightarrow \psi\|_{\mathbf{M},v_1} \leq \|\varphi \Rightarrow \psi\|_{\mathbf{M},v_2}.$$

Since $\|\varphi \Rightarrow \psi\|_{\mathbf{M},v_2} = \|\varphi\|_{\mathbf{M},v_2} \to \|\varphi\|_{\mathbf{M},v_2}$, adjointness yields that the required inequality is equivalent to

$$\left(\bigotimes_{x\in \mathrm{free}(\varphi\Rightarrow\psi)} (v_1(x) \approx^{\mathbf{M}} v_2(x))^{|\varphi\Rightarrow\psi|_x}\right) \otimes \|\varphi \Rightarrow \psi\|_{\mathbf{M},v_1} \otimes \|\varphi\|_{\mathbf{M},v_2} \quad \leq \|\psi\|_{\mathbf{M},v_2},$$

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which is true. Indeed,

$$\left(\bigotimes_{x\in \mathrm{free}(\varphi\Rightarrow\psi)}(v_1(x)\approx^{\mathbf{M}} v_2(x))^{|\varphi\Rightarrow\psi|_x}\right)\otimes\|\varphi\Rightarrow\psi\|_{\mathbf{M},v_1}\otimes\|\varphi\|_{\mathbf{M},v_2}$$

$$= \left(\bigotimes_{x\in \mathrm{free}(\psi)}(v_1(x)\approx^{\mathbf{M}} v_2(x))^{|\psi|_x}\right)\otimes\left(\bigotimes_{x\in \mathrm{free}(\varphi)}(v_1(x)\approx^{\mathbf{M}} v_2(x))^{|\varphi|_x}\right)$$

$$\otimes\|\varphi\|_{\mathbf{M},v_2}\otimes(\|\varphi\|_{\mathbf{M},v_1}\to\|\psi\|_{\mathbf{M},v_1})$$

$$\le \left(\bigotimes_{x\in \mathrm{free}(\psi)}(v_1(x)\approx^{\mathbf{M}} v_2(x))^{|\psi|_x}\right)\otimes\|\varphi\|_{\mathbf{M},v_1}\otimes(\|\varphi\|_{\mathbf{M},v_1}\to\|\psi\|_{\mathbf{M},v_1})$$

$$\le \left(\bigotimes_{x\in \mathrm{free}(\psi)}(v_1(x)\approx^{\mathbf{M}} v_2(x))^{|\psi|_x}\right)\otimes\|\psi\|_{\mathbf{M},v_1}\le\|\psi\|_{\mathbf{M},v_2}.$$

Consider now quantifiers.

$$\left(\bigotimes_{x\in \mathrm{free}((\forall y)\varphi)}(v_1(x)\approx^{\mathbf{M}} v_2(x))^{|(\forall y)\varphi|_x}\right)\otimes\|(\forall y)\varphi\|_{\mathbf{M},v_1}$$

$$= \left(\bigotimes_{x\in \mathrm{free}((\forall y)\varphi)}(v_1(x)\approx^{\mathbf{M}} v_2(x))^{|\varphi|_x}\right)\otimes\bigwedge_{v_1'=_y v_1}\|\varphi\|_{\mathbf{M},v_1'}$$

$$\le \bigwedge_{v_1'=_y v_1}\left(\left(\bigotimes_{x\in \mathrm{free}((\forall y)\varphi)}(v_1(x)\approx^{\mathbf{M}} v_2(x))^{|\varphi|_x}\right)\otimes\|\varphi\|_{\mathbf{M},v_1'}\right)$$

$$= \bigwedge_{v_1'=_y v_1}\left(\left(\bigotimes_{x\in \mathrm{free}((\forall y)\varphi)}(v_1'(x)\approx^{\mathbf{M}} v_2'(x))^{|\varphi|_x}\right)\otimes\|\varphi\|_{\mathbf{M},v_1'}\right)\le\|\varphi\|_{\mathbf{M},v_2'}$$

for any valuation v_2' such that $v_2' =_y v_2$. Therefore, we also have

$$\left(\bigotimes_{x\in \mathrm{free}((\forall y)\varphi)}(v_1(x)\approx^{\mathbf{M}} v_2(x))^{|(\forall y)\varphi|_x}\right)\otimes\|(\forall y)\varphi\|_{\mathbf{M},v_1}\le\bigwedge_{v_2'=_y v_2}\|\varphi\|_{\mathbf{M},v_2'}=\|(\forall y)\varphi\|_{\mathbf{M},v_2}.$$

We used the obvious fact that if $v_1 =_y v_1'$ and $v_2 =_y v_2'$, then $(v_1(x)\approx^{\mathbf{M}} v_2(x)) = (v_1'(x)\approx^{\mathbf{M}} v_2'(x))$ for any $x\in\mathrm{free}((\forall y)\varphi)$. Thus, the assertion is valid for $(\forall y)\varphi$.

Finally, for $(\exists y)\varphi$ we similarly have

$$\left(\bigotimes_{x\in \mathrm{free}((\exists y)\varphi)}(v_1(x)\approx^{\mathbf{M}} v_2(x))^{|(\exists y)\varphi|_x}\right)\otimes\|(\exists y)\varphi\|_{\mathbf{M},v_1}$$

$$= \left(\bigotimes_{x\in \mathrm{free}((\exists y)\varphi)}(v_1(x)\approx^{\mathbf{M}} v_2(x))^{|\varphi|_x}\right)\otimes\bigvee_{v_1'=_y v_1}\|\varphi\|_{\mathbf{M},v_1'}$$

$$= \bigvee_{v_1'=_y v_1}\left(\left(\bigotimes_{x\in \mathrm{free}((\exists y)\varphi)}(v_1(x)\approx^{\mathbf{M}} v_2(x))^{|\varphi|_x}\right)\otimes\|\varphi\|_{\mathbf{M},v_1'}\right)$$

$$= \bigvee_{v_1'=_y v_1}\left(\left(\bigotimes_{x\in \mathrm{free}((\exists y)\varphi)}(v_1'(x)\approx^{\mathbf{M}} (v_2'(v_1'))(x))^{|\varphi|_x}\right)\otimes\|\varphi\|_{\mathbf{M},v_1'}\right)$$

$$\le \bigvee_{v_1'=_y v_1}\|\varphi\|_{\mathbf{M},v_2'(v_1')}=\bigvee_{v_2'=_y v_2}\|\varphi\|_{\mathbf{M},v_2'}=\|(\exists y)\varphi\|_{\mathbf{M},v_2},$$

where $v_2'(v_1')$ is a valuation such that $v_2'(v_1') =_y v_2$ and $(v_2'(v_1'))(y) = v_1'(y)$. □

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For **L**-structures $\mathbf{M}_1$ and $\mathbf{M}_2$ with a common universe $M = M_1 = M_2$ and a common fuzzy equivalence $\approx^{\mathbf{M}}$, and for $(f^{\mathbf{M}_1} \approx f^{\mathbf{M}_2})$ defined by (12) we have the following lemma.

LEMMA 5. Let $\mathbf{M}_1, \mathbf{M}_2$ be **L**-structures of type $\|R, F, \sigma\|$ that have the same support set M, t be a term, v be a valuation. Then we have

$$\bigotimes_{f\in F}\left(f^{\mathbf{M}_1} \approx f^{\mathbf{M}_2}\right)^{|t|_f} \leq \|t\|_{\mathbf{M}_1,v} \approx^{\mathbf{M}} \|t\|_{\mathbf{M}_2,v}.$$

Proof. We prove the assertion by induction over complexity of t. If t is a variable, then the assertion is trivial. For the induction step, let $t = g(t_1, \ldots, t_n)$ and assume the assertion is valid for $t_1, \ldots, t_n$. We have

$$\begin{aligned}
&\bigotimes_{f\in F}\left(f^{\mathbf{M}_1} \approx f^{\mathbf{M}_2}\right)^{|t|_f}\\
&= \left(g^{\mathbf{M}_1} \approx g^{\mathbf{M}_2}\right)^{|t|_g} \otimes \bigotimes_{f\neq g}\left(f^{\mathbf{M}_1} \approx f^{\mathbf{M}_2}\right)^{|t|_f}\\
&= \left(g^{\mathbf{M}_1} \approx g^{\mathbf{M}_2}\right)^{1+\sum_{i=1}^{n}|t_i|_g} \otimes \bigotimes_{f\neq g, |t|_f}\left(f^{\mathbf{M}_1} \approx f^{\mathbf{M}_2}\right)^{\sum_{i=1}^{n}|t_i|_f}\\
&= \left(g^{\mathbf{M}_1} \approx g^{\mathbf{M}_2}\right) \otimes \bigotimes_{i=1}^{n} \bigotimes_{f\in F, |t_i|_f>0}\left(f^{\mathbf{M}_1} \approx f^{\mathbf{M}_2}\right)^{|t_i|_f}\\
&\leq \left(g^{\mathbf{M}_1} \approx g^{\mathbf{M}_2}\right) \otimes \bigotimes_{i=1}^{n}\left(\|t_i\|_{\mathbf{M}_1,v} \approx^{\mathbf{M}} \|t_i\|_{\mathbf{M}_2,v}\right)\\
&\leq \left(g^{\mathbf{M}_1} \approx g^{\mathbf{M}_2}\right) \otimes \left(g^{\mathbf{M}_2}\left(\|t_1\|_{\mathbf{M}_1,v}, \ldots, \|t_n\|_{\mathbf{M}_1,v}\right)\right.\\
&\qquad \left.\approx^{\mathbf{M}} \left(g^{\mathbf{M}_2}\left(\|t_1\|_{\mathbf{M}_2,v}, \ldots, \|t_n\|_{\mathbf{M}_2,v}\right)\right)\right.\\
&= \left(\bigwedge_{m_1,\ldots,m_n} g^{\mathbf{M}_1}(m_1, \ldots, m_n) \approx^{\mathbf{M}} g^{\mathbf{M}_2}(m_1, \ldots, m_n)\right)\\
&\qquad \otimes\left(g^{\mathbf{M}_2}\left(\|t_1\|_{M_1,v}, \ldots, \|t_n\|_{\mathbf{M}_1,v}\right) \approx^{\mathbf{M}} g^{\mathbf{M}_2}\left(\|t_1\|_{\mathbf{M}_2,v}, \ldots, \|t_n\|_{\mathbf{M}_2,v}\right)\right)\\
&\leq \left(g^{\mathbf{M}_1}\left(\|t_1\|_{\mathbf{M}_1,v}, \ldots, \|t_n\|_{\mathbf{M}_1,v}\right) \approx^{\mathbf{M}} g^{\mathbf{M}_2}\left(\|t_1\|_{\mathbf{M}_1,v}, \ldots, \|t_n\|_{\mathbf{M}_1,v}\right)\right)\\
&\qquad \otimes\left(g^{\mathbf{M}_2}\left(\|t_1\|_{\mathbf{M}_1,v}, \ldots, \|t_n\|_{\mathbf{M}_1,v}\right) \approx^{\mathbf{M}} g^{\mathbf{M}_2}\left(\|t_1\|_{\mathbf{M}_2,v}, \ldots, \|t_n\|_{\mathbf{M}_2,v}\right)\right)\\
&\leq g^{\mathbf{M}_1}\left(\|t_1\|_{\mathbf{M}_1,v}, \ldots, \|t_n\|_{\mathbf{M}_1,v}\right) \approx^{\mathbf{M}} g^{\mathbf{M}_2}\left(\|t_1\|_{\mathbf{M}_2,v}, \ldots, \|t_n\|_{\mathbf{M}_2,v}\right)\\
&= \|t\|_{\mathbf{M}_1,v} \approx^{\mathbf{M}} \|t\|_{\mathbf{M}_2,v},
\end{aligned}$$

completing the proof. □

Lemma 5 gives an estimation of similarity $\|t\|_{\mathbf{M}_1,v} \approx^{\mathbf{M}} \|t\|_{\mathbf{M}_2,v}$ of values of a term interpreted by a valuation in two fuzzy structures with the same support. An analogous result for estimation of $\|\varphi\|_{\mathbf{M}_1,v} \leftrightarrow \|\varphi\|_{\mathbf{M}_2,v}$ is presented in the next lemma.

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LEMMA 6. Let $\mathbf{M}_1$, $\mathbf{M}_2$ be $\mathbf{L}$-structures of type $\|R, F, \sigma\|$ with a common universe M, φ be a formula, v be a valuation. Then

$$\bigotimes_{r\in R}\left(r^{\mathbf{M}_1} \approx r^{\mathbf{M}_2}\right)^{|\varphi|_r}\otimes\bigotimes_{f\in F}\left(f^{\mathbf{M}_1} \approx f^{\mathbf{M}_2}\right)^{|\varphi|_f}\leq \|\varphi\|_{\mathbf{M}_1,v} \leftrightarrow \|\varphi\|_{\mathbf{M}_2,v}.$$

Proof. It is easy to see that we have to prove

$$\bigotimes_{r\in R}\left(r^{\mathbf{M}_1} \approx r^{\mathbf{M}_2}\right)^{|\varphi|_r}\otimes\bigotimes_{f\in F}\left(f^{\mathbf{M}_1} \approx f^{\mathbf{M}_2}\right)^{|\varphi|_f}\otimes\|\varphi\|_{\mathbf{M}_1,v} \leq \|\varphi\|_{\mathbf{M}_2,v}.$$

The proof goes by induction over the complexity of φ. If $\varphi = s(t_1, \ldots, t_m)$ then, by Lemma 5,

$$\begin{aligned}
&\bigotimes_{r\in R}\left(r^{\mathbf{M}_1} \approx r^{\mathbf{M}_2}\right)^{|\varphi|_r}\otimes\bigotimes_{f\in F}\left(f^{\mathbf{M}_1} \approx f^{\mathbf{M}_2}\right)^{|\varphi|_f}\\
&= \left(s^{\mathbf{M}_1} \approx s^{\mathbf{M}_2}\right)\otimes\bigotimes_{f\in F}\left(f^{\mathbf{M}_1} \approx f^{\mathbf{M}_2}\right)^{|t_1|_f+\cdots+|t_m|_f}\\
&= \left(s^{\mathbf{M}_1} \approx s^{\mathbf{M}_2}\right)\otimes\bigotimes_{i=1}^{m}\bigotimes_{f\in F}\left(f^{\mathbf{M}_1} \approx f^{\mathbf{M}_2}\right)^{|t_i|_f}\\
&\leq \left(s^{\mathbf{M}_1} \approx s^{\mathbf{M}_2}\right)\otimes\bigotimes_{i=1}^{m}\left(\|t_i\|_{\mathbf{M}_1,v} \approx^{\mathbf{M}_2} \|t_i\|_{\mathbf{M}_2,v}\right)\\
&\leq \left(s^{\mathbf{M}_1} \approx s^{\mathbf{M}_2}\right)\otimes\left(s^{\mathbf{M}_2}\left(\|t_1\|_{\mathbf{M}_1,v}, \ldots, \|t_m\|_{\mathbf{M}_1,v}\right) \leftrightarrow s^{\mathbf{M}_2}(\|t_1\|_{\mathbf{M}_2,v}, \ldots, \|t_m\|_{\mathbf{M}_2,v})\right)\\
&\leq \left(s^{\mathbf{M}_1}(\|t_1\|_{\mathbf{M}_1,v}, \ldots, \|t_m\|_{\mathbf{M}_1,v}) \leftrightarrow s^{\mathbf{M}_2}(\|t_1\|_{\mathbf{M}_1,v}, \ldots, \|t_m\|_{\mathbf{M}_1,v})\right)\\
&\quad\otimes\left(s^{\mathbf{M}_2}\left(\|t_1\|_{\mathbf{M}_1,v}, \ldots, \|t_m\|_{\mathbf{M}_1,v}\right) \leftrightarrow s^{\mathbf{M}_2}\left(\|t_1\|_{\mathbf{M}_2,v}, \ldots, \|t_m\|_{\mathbf{M}_2,v}\right)\right)\\
&\leq \left(s^{\mathbf{M}_1}\left(\|t_1\|_{\mathbf{M}_1,v}, \ldots, \|t_1\|_{\mathbf{M}_1,v}\right) \leftrightarrow s^{\mathbf{M}_2}\left(\|t_1\|_{\mathbf{M}_2,v}, \ldots, \|t_1\|_{\mathbf{M}_2,v}\right)\right)\\
&= \|\varphi\|_{\mathbf{M}_1,v} \leftrightarrow \|\varphi\|_{\mathbf{M}_2,v}.
\end{aligned}$$

Let the assertion be valid for φ and ψ. We have

$$\begin{aligned}
&\bigotimes_{r\in R}(r^{\mathbf{M}_1} \approx r^{\mathbf{M}_2})^{|\varphi\wedge\psi|_r} \otimes \bigotimes_{f\in F}(f^{\mathbf{M}_1} \approx f^{\mathbf{M}_2})^{|\varphi\wedge\psi|_f} \otimes \|\varphi \wedge \psi\|_{\mathbf{M}_1,v}\\
&\quad= \bigotimes_{r\in R}(r^{\mathbf{M}_1} \approx r^{\mathbf{M}_2})^{|\varphi\wedge\psi|_r} \otimes \bigotimes_{f\in F}(f^{\mathbf{M}_1} \approx f^{\mathbf{M}_2})^{|\varphi\wedge\psi|_f}\\
&\qquad\otimes(\|\varphi\|_{\mathbf{M}_1,v} \wedge \|\psi\|_{\mathbf{M}_1,v})\\
&\quad\leq (\bigotimes_{r\in R}(r^{\mathbf{M}_1} \approx r^{\mathbf{M}_2})^{|\varphi|_r} \otimes \bigotimes_{f\in F}(f^{\mathbf{M}_1} \approx f^{\mathbf{M}_2})^{|\varphi|_f} \otimes \|\varphi\|_{\mathbf{M}_1,v})\\
&\qquad\wedge(\bigotimes_{r\in R}(r^{\mathbf{M}_1} \approx r^{\mathbf{M}_2})^{|\psi|_r} \otimes \bigotimes_{f\in F}(f^{\mathbf{M}_1} \approx f^{\mathbf{M}_2})^{|\psi|_f} \otimes \|\psi\|_{\mathbf{M}_1,v})\\
&\quad\leq \|\varphi\|_{\mathbf{M}_2,v} \wedge \|\psi\|_{\mathbf{M}_2,v} = \|\varphi \wedge \psi\|_{\mathbf{M}_2,v}.
\end{aligned}$$

The proof for $\varphi\vee\psi$ can be done in the same way as for $\varphi\wedge\psi$ (just replace $\wedge$ by $\vee$).

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Consider $\varphi \otimes \psi$:

$$\begin{aligned}
& \bigotimes_{r\in R}(r^{\mathbf{M}_1} \approx r^{\mathbf{M}_2})^{|\varphi\otimes\psi|_r} \otimes \bigotimes_{f\in F}(f^{\mathbf{M}_1} \approx f^{\mathbf{M}_2})^{|\varphi\otimes\psi|_f} \otimes \|\varphi \otimes \psi\|_{\mathbf{M}_1,v} = \\
= \quad & \bigotimes_{r\in R}(r^{\mathbf{M}_1} \approx r^{\mathbf{M}_2})^{|\varphi|_r} \otimes \bigotimes_{f\in F}(f^{\mathbf{M}_1} \approx f^{\mathbf{M}_2})^{|\varphi|_f} \otimes \|\varphi\|_{\mathbf{M}_1,v} \otimes \\
& \otimes \bigotimes_{r\in R}(r^{\mathbf{M}_1} \approx r^{\mathbf{M}_2})^{|\varphi|_r} \otimes \bigotimes_{f\in F}(f^{\mathbf{M}_1} \approx f^{\mathbf{M}_2})^{|\varphi|_f} \otimes \|\psi\|_{\mathbf{M}_1,v} \leq \\
\leq \quad & \|\varphi\|_{\mathbf{M}_2,v} \otimes \|\psi\|_{\mathbf{M}_2,v} = \|\varphi \otimes \psi\|_{\mathbf{M}_2,v}.
\end{aligned}$$

For $\varphi \Rightarrow \psi$, adjointness implies that we have to show

$$\bigotimes_{r\in R}\left(r^{\mathbf{M}_1} \approx r^{\mathbf{M}_2}\right)^{|\varphi\Rightarrow\psi|_r} \otimes \bigotimes_{f\in F}\left(f^{\mathbf{M}_1} \approx f^{\mathbf{M}_2}\right)^{|\varphi\Rightarrow\psi|_f} \quad \otimes \|\varphi \Rightarrow \psi\|_{\mathbf{M}_1,v} \otimes \|\varphi\|_{\mathbf{M}_2,v} \leq \|\psi\|_{\mathbf{M}_2,v}.$$

This inequality is true:

$$\begin{aligned}
& \bigotimes_{r\in R}\left(r^{\mathbf{M}_1} \approx r^{\mathbf{M}_2}\right)^{|\varphi\Rightarrow\psi|_r} \otimes \bigotimes_{f\in F}\left(f^{\mathbf{M}_1} \approx f^{\mathbf{M}_2}\right)^{|\varphi\Rightarrow\psi|_f} \quad \otimes \|\varphi \Rightarrow \psi\|_{\mathbf{M}_1,v} \otimes \|\varphi\|_{\mathbf{M}_2,v} \\
& \quad = \bigotimes_{r\in R}\left(r^{\mathbf{M}_1} \approx r^{\mathbf{M}_2}\right)^{|\varphi|_r+|\psi|_r} \otimes \bigotimes_{f\in F}\left(f^{\mathbf{M}_1} \approx f^{\mathbf{M}_2}\right)^{|\varphi|_f+|\psi|_f} \otimes \|\varphi\|_{\mathbf{M}_2,v} \otimes (\|\varphi\|_{\mathbf{M}_1,v} \to \|\psi\|_{\mathbf{M}_1,v}) \\
& \quad \leq \bigotimes_{r\in R}\left(r^{\mathbf{M}_1} \approx r^{\mathbf{M}_2}\right)^{|\psi|_r} \otimes \bigotimes_{f\in F}\left(f^{\mathbf{M}_1} \approx f^{\mathbf{M}_2}\right)^{|\psi|_f} \otimes \|\varphi\|_{\mathbf{M}_1,v} \otimes \left(\|\varphi\|_{\mathbf{M}_1,v} \to \|\psi\|_{\mathbf{M}_1,v}\right) \\
& \quad \leq \bigotimes_{r\in R}\left(r^{\mathbf{M}_1} \approx r^{\mathbf{M}_2}\right)^{|\psi|_r} \otimes \bigotimes_{f\in F}\left(f^{\mathbf{M}_1} \approx f^{\mathbf{M}_2}\right)^{|\psi|_f} \otimes \|\psi\|_{\mathbf{M}_1,v} \leq \|\psi\|_{\mathbf{M}_1,v}.
\end{aligned}$$

For $(\forall y)\varphi$ we have

$$\begin{aligned}
& \bigotimes_{r\in R}\left(r^{\mathbf{M}_1} \approx r^{\mathbf{M}_2}\right)^{|(\forall y)\varphi|_r} \otimes \bigotimes_{f\in F}\left(f^{\mathbf{M}_1} \approx f^{\mathbf{M}_2}\right)^{|(\forall y)\varphi|_f} \otimes \|(\forall y)\varphi\|_{\mathbf{M}_1,v} \\
& \quad = \bigotimes_{r\in R}\left(r^{\mathbf{M}_1} \approx r^{\mathbf{M}_2}\right)^{|\varphi|_r} \otimes \bigotimes_{f\in F}\left(f^{\mathbf{M}_1} \approx f^{\mathbf{M}_2}\right)^{|\varphi|_f} \otimes \bigwedge_{v'=_y v} \|\varphi\|_{\mathbf{M}_1,v'} \\
& \quad \leq \bigwedge_{v'=_y v} \left(\bigotimes_{r\in R}\left(r^{\mathbf{M}_1} \approx r^{\mathbf{M}_2}\right)^{|\varphi|_r} \otimes \bigotimes_{f\in F}\left(f^{\mathbf{M}_1} \approx f^{\mathbf{M}_2}\right)^{|\varphi|_f} \otimes \|\varphi\|_{\mathbf{M}_1,v'} \right) \\
& \quad \leq \bigwedge_{v'=_y v} \|\varphi\|_{\mathbf{M}_2,v'} = \|(\forall y)\varphi\|_{\mathbf{M}_2,v}.
\end{aligned}$$

For $(\exists y)\varphi$, the above proof for $(\forall y)\varphi$ works as well (just replace $\forall$ by $\exists$). □

Now, Theorem 1 and Theorem 2 follow as easy consequences.

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Proof of Theorem 1. By Lemma 4 and Lemma 6 we get:

$$\bigotimes_{x\in \mathrm{free}(\varphi)}(v_1(x) \approx^{\mathbf{M}} v_2(x))^{|\varphi|_x} \otimes \bigotimes_{r\in R}\left(r^{\mathbf{M}_1} \approx r^{\mathbf{M}_2}\right)^{|\varphi|_r} \otimes \bigotimes_{f\in F}\left(f^{\mathbf{M}_1} \approx f^{\mathbf{M}_2}\right)^{|\varphi|_f}$$

$$\leq (\|\varphi\|_{\mathbf{M}_1,v_1} \leftrightarrow \|\varphi\|_{\mathbf{M}_1,v_2}) \otimes (\|\varphi\|_{\mathbf{M}_1,v_2} \leftrightarrow \|\varphi\|_{\mathbf{M}_2,v_2}) \leq \|\varphi\|_{\mathbf{M}_1,v_1} \leftrightarrow \|\varphi\|_{\mathbf{M}_2,v_2}.$$

Proof of Theorem 2. By Lemma 3 and Lemma 5 we have

$$\bigotimes_{x\in \mathrm{var}(t)}(v_1(x) \approx^{\mathbf{M}} v_2(x))^{|t|_x} \otimes \bigotimes_{f\in F}\left(f^{\mathbf{M}_1} \approx f^{\mathbf{M}_2}\right)^{|t|_f}$$

$$\leq \left(\|t\|_{\mathbf{M}_1,v_1} \approx^{\mathbf{M}} \|t\|_{\mathbf{M}_1,v_2}\right) \otimes \left(\|t\|_{\mathbf{M}_1,v_2} \approx^{\mathbf{M}} \|t\|_{\mathbf{M}_2,v_2}\right) \leq \|t\|_{\mathbf{M}_1,v_1} \approx^{\mathbf{M}} \|t\|_{\mathbf{M}_2,v_2},$$

proving the assertion. □