INS 6610
DISK / 30/3/02
ARTICLE IN PRESS
No. of pages: 5
DTD 4.3.1/SPS

ELSEVIER
Information Sciences xxx (2002) xxx–xxx
INFORMATION SCIENCES
AN INTERNATIONAL JOURNAL
www.elsevier.com/locate/ins

Determinism and fuzzy automata

Radim Bělohlávek

Department of Computer Science, Palacký University of Olomouc, Czech Republic
Institute for Research and Applications of Fuzzy Modeling, University of Ostrava, 30, dubna 7, Ostrava, Czech Republic

Received 23 October 2000; received in revised form 20 September 2001; accepted 21 November 2001

Abstract

The concept of a fuzzy automaton, as studied in the literature, is a direct generalization of the classical concept of a non-deterministic automaton. In this paper, we consider the question of whether there is a deterministic counterpart to the notion of a fuzzy automaton. We propose a definition of deterministic fuzzy automata and show that they are equally powerful as fuzzy automata. Compared to the classical notion of a deterministic automaton, the deterministic fuzzy automaton defined in our paper differs essentially only in that the final states form a fuzzy set. Moreover, we show that deterministic fuzzy automata are equally powerful as nested systems of deterministic automata. © 2002 Published by Elsevier Science Inc.

Keywords: Automata; Fuzzy sets; Fuzzy automata; Deterministic automata

1. Introduction and preliminaries

Fuzzy approach (graded truth approach) has been applied to automata theory in the very early age of fuzzy set theory (see e.g. [4,5] or [1] for further references). There is a deep reason to study fuzzy automata: several languages are fuzzy by nature (e.g. the language containing words in which many letters "a" occur).

E-mail address: radim.belohlavek@upol.cz (R. Bělohlávek).

PII: S0020-0255(02)00192-5

INS 6610 **ARTICLE IN PRESS** No. of pages: 5
DISK / 30/3/02 DTD 4.3.1/SPS

In our paper we consider finite automata. The basic idea in the formulation of a fuzzy automaton is that, unlike the classical case, the automaton can switch from one state to another one to a certain (truth) degree. For our purpose we adopt the following definition of a fuzzy automaton.

We will use complete lattices as the structures of truth values. Note that usually the real unit interval $[0,1]$ is used in the literature (the reader not familiar with lattices may, without any harm, substitute $[0,1]$ for complete lattices throughout the paper). A complete lattice with a support set L and the lattice order $\leqslant$ will be denoted by $\mathbf{L} = \langle L, \leqslant \rangle$. The only requirement put on complete lattices that will be assumed throughout the paper is that for any finite subset L' of L, the complete sublattice $[L']$ of L generated by L' be finite (this requirement is trivially satisfied by $[0,1]$ or by any chain). Infima and suprema are denoted by $\bigwedge$ and $\bigvee$ etc. A fuzzy set in a universe set X with truth values in a complete lattice $\mathbf{L}$ is any mapping $A : X \to L$. To make the structure of truth values apparent, we write also **L**-set instead of fuzzy set (and, similarly, in general, instead of fuzzy ... we write **L**-... etc.). The set of all **L**-sets in X is denoted by L^X. Note that $\{a/x\}$ denotes the fuzzy set A (so-called singleton) in X such that $A(x) = a$ and $A(y) = 0$ for $y \neq x$.

For a set X, X^* denotes the set of all finite sequencies (words) of elements of X including the empty word ε. For $w \in X^*$, $|w|$ denotes the length of w (note that $|\varepsilon| = 0$).

A *fuzzy automaton* (FA) is a tuple $\mathscr{A} = \langle S, \Sigma, F, P, G \rangle$ where S is a finite set (of states), Σ is a finite set (input alphabet), $P \in L^S$ and $G \in L^S$ are **L**-sets of initial and final states, respectively, and $F \in L^{S\times\Sigma\times S}$ is a fuzzy relation between S, Σ, and S (transition relation). For $\sigma \in \Sigma$, denote $F_\sigma \in L^{S\times S}$ by

$$F_\sigma(s_1, s_2) = F(s_1, \sigma, s_2).$$

The degree $(L(\mathscr{A}))(w)$ to which a fuzzy automaton $\mathscr{A}$ accepts a word $\sigma_1 \cdots \sigma_n \in \Sigma^*$ is defined by

$$(L(\mathscr{A}))(\sigma_1 \cdots \sigma_n) = P \circ F_{\sigma_1} \circ \cdots \circ F_{\sigma_n} \circ G,$$

where $\circ$ denotes the max–min composition of fuzzy relation, i.e.

$$(L(\mathscr{A}))(\sigma_1 \cdots \sigma_n) = \bigvee_{s_1,\ldots,s_{n+1} \in S} P(s_1) \wedge F_{\sigma_1}(s_1, s_2) \wedge \cdots \wedge F_{\sigma_n}(s_n, s_{n+1}) \wedge G(s_{n+1}).$$

This **L**-set, $L(\mathscr{A})$, in Σ^* is called the fuzzy language accepted by $\mathscr{A}$.

2. Deterministic fuzzy automata

The notion of a fuzzy automaton is a generalization of the notion of a nondeterministic automaton: instead of sets of initial and final states we have fuzzy sets of initial and final states; instead of a (bivalent) transition relation we have fuzzy transition relation. Note that, usually, the transition of non-deterministic automata is defined by a so-called transition function assigning to each state s and each input symbol σ a subset $\delta(s,\sigma)$ of states to which the automaton can go. This is, however, only a matter of taste: we could equivalently define a fuzzy automaton using the transition function assigning to a state s and an input symbol σ a fuzzy set $\delta(s,\sigma)$ of states (the relationship to our transition relation would be $(\delta(s,\sigma)(s') = F(s,\sigma,s'))$.

The fuzzy automaton is non-deterministic in nature: there may be non-zero truth degrees that the automaton can go to more than one state (given a state and an input symbol). In the following we are going to present a deterministic counterpart of the notion of a fuzzy automaton.

A *deterministic fuzzy automaton* (DFA) is a tuple $\mathscr{A} = \langle S, \Sigma, \delta, s_0, G\rangle$ where S is a finite set (of states), Σ is a finite set (of input symbols), $s_0 \in S$ (input symbol), $G \in L^S$ is a fuzzy set of final states, and $\delta : S \times \Sigma \to S$ is a function (transition function).

Transition function δ may be extended to $\delta^* : S \times \Sigma^* \to S$ by putting $\delta^*(s,\varepsilon) = s$ and $\delta^*(s, \sigma_1\sigma_2\cdots\sigma_n) = \delta^*(\delta(s,\sigma_1), \sigma_2\cdots\sigma_n)$. The degree $(L(\mathscr{A}))(\sigma_1\cdots\sigma_n)$ to which a deterministic fuzzy automaton $\mathscr{A}$ accepts a word $\sigma_1\cdots\sigma_n \in \Sigma^*$ is defined by

$$(L(\mathscr{A}))(\sigma_1\cdots\sigma_n) = G(\delta^*(s_0, \sigma_1\cdots\sigma_n)).$$

The **L**-set $L(\mathscr{A})$ is called the fuzzy language accepted by $\mathscr{A}$.

Note that our definition differs from the usual definition of a deterministic automaton only in that the final states form a fuzzy set. This, however, makes it possible to accept words to certain truth degrees, and thus to recognize fuzzy languages. In fact, the following theorem shows that fuzzy automata and non-deterministic automata are equally powerful.

Theorem 2.1. *A fuzzy language is accepted by some fuzzy automaton iff it is accepted by some deterministic fuzzy automaton.*

Proof. If a fuzzy language is accepted by a DFA $\mathscr{A} = \langle S, \Sigma, \delta, s_0, G\rangle$, then it is easy to see that it is accepted by the FA $\mathscr{B} = \langle S, \Sigma, F, \{1/s_0\}, G\rangle$ where $F(s_1,\sigma,s_2) = 1$ if $\delta(s_1,\sigma) = s_2$ and $F(s_1,\sigma,s_2) = 0$ otherwise.

Conversely, let a fuzzy language be accepted by a fuzzy automaton $\mathscr{A} = \langle S, \Sigma, F, P, G\rangle$. Let $M = [L']$ be the complete sublattice of L generated by

$$L' = \{F(s_1,\sigma,s_2) \mid s_1, s_2 \in S, \sigma \in \Sigma\} \cup \{P(s) \mid s \in S\}$$

and let $0, 1 \in M$. Since S and Σ are finite, L' is finite, and by assumption, M is finite. Define a DFA $\mathscr{B} = \{S', \Sigma, \delta, s_0, G'\}$ by

$S' = M^S$ (i.e. S' is the set of all **L**-sets in S that have truth values in M),

$\delta(s,\sigma) = s \circ F_\sigma$ for $s \in S', \sigma \in \Sigma$,

INS 6610 **ARTICLE IN PRESS** No. of pages: 5
DISK / 30/3/02 DTD 4.3.1/SPS

$s_0 = \{1/P\}$,

$G'(s) = s \circ G$ for $s \in S'$.

Notice first that the above definition is correct: S' is finite and $\delta(s,\sigma) \in S'$ (this follows from the fact that M is closed w.r.t. infima and suprema and that $(\delta(s,\sigma))(s') = \bigvee_{s'' \in S} s(s'') \wedge F_\sigma(s'', s'))$. We have to prove that $L(\mathscr{A}) = L(\mathscr{B})$.

Observe first that for any word $\sigma_1 \cdots \sigma_n$ and any $s \in S'$ we have

$$\delta^*(s, \sigma_1 \cdots \sigma_n) = s \circ F_{\sigma_1} \circ \cdots \circ F_{\sigma_n}.$$

Indeed, by induction: for $n = 0$ (the empty word) we have $\delta^*(s, \varepsilon) = s$. Assuming that the assertion holds for $n-1$, we get $\delta^*(s, \sigma_1 \cdots \sigma_n) = \delta^*(\delta(s, \sigma_1), \sigma_2 \cdots \sigma_n) = \delta(s, \sigma_1) \circ F_{\sigma_2} \circ \cdots \circ F_{\sigma_n} = s \circ F_{\sigma_1} \circ F_{\sigma_2} \circ \cdots \circ F_{\sigma_n}$.

Now, we have

$$(L(\mathscr{A}))(\sigma_1 \cdots \sigma_n) = P \circ F_{\sigma_1} \circ \cdots \circ F_{\sigma_n} \circ G$$

and

$$\begin{aligned}(L(\mathscr{B}))(\sigma_1 \cdots \sigma_n) &= G'(\delta^*(s_0, \sigma_1 \cdots \sigma_n)) = G'(P \circ F_{\sigma_1} \circ \cdots \circ F_{\sigma_1}) \\ &= P \circ F_{\sigma_1} \circ \cdots \circ F_{\sigma_n} \circ G,\end{aligned}$$

proving the assertion. □

Note that if $L = [0,1]$ (or, more generally, L is a chain) then M is directly equal to $L' \cup \{0,1\}$.

The concept of a deterministic fuzzy automaton differs from the concept of a deterministic automaton only in that the final states form a fuzzy set. It is natural to ask whether deterministic fuzzy automata can be in some natural way represented by deterministic automata. Note that the question has been affirmatively answered for fuzzy automata and non-deterministic (non-fuzzy) automata in [2].

Recall that an a-cut ($a \in L$) of an **L**-set A in X is a subset ${}^aA$ of X defined by ${}^aA = \{x \in X \mid A(x) \geqslant a\}$. Let us call a system $\{A_a \subseteq X \mid a \in L\}$ **L**-nested if

(i) $a \leqslant b$ implies $A_b \subseteq A_a$ (for any $a, b \in L$),

(ii) $A_0 = X$,

(iii) for each $x \in X$, the set $\{a \in L \mid x \in A_a\}$ has the greatest element.

A system $\{\mathscr{A}_a \mid a \in L\}$ of deterministic automata $\mathscr{A}_a = \{S, \Sigma, \delta, s_0, G_a\}$ (notice that the set of states, the alphabet, the transition function, and the initial state are common) is called **L**-nested if $\{G_a \mid a \in L\}$ is an **L**-nested system of subsets of S. For an **L**-nested system $\mathbf{S} = \{\mathscr{A}_a \mid a \in L\}$ of deterministic automata we define the degree to which **S** accepts a word $w \in \Sigma^*$ by

$$L(\mathbf{S})(w) = \bigvee \{a \in L \mid \mathscr{A}_a \text{ accepts } w\}.$$

The thus-defined **L**-set $L(\mathbf{S})$ in Σ^* is called the fuzzy language accepted by **S**. **L**-nested systems of deterministic automata are equally powerful as deterministic fuzzy automata.

INS 6610 ARTICLE IN PRESS No. of pages: 5
DISK / 30/3/02 DTD 4.3.1/SPS

Theorem 2.2. *A fuzzy language is accepted by some deterministic fuzzy automaton iff it is accepted by some* **L**-*nested system of deterministic automata.*

Proof. Let $\mathscr{A} = \langle S, \Sigma, \delta, s_0, G\rangle$ be a deterministic fuzzy automaton. Put $\mathscr{A}_a = \{S, \Sigma, \delta, s_0, {}^aG\}$ and consider the system $\mathbf{S} = \{\mathscr{A}_a \mid a \in L\}$. It is obvious that $\mathbf{S}$ is **L**-nested. By definition, we have to check that $\bigvee\{a \in L \mid \mathscr{A}_a$ accepts $w\} = L(\mathscr{A})$, i.e. $\bigvee\{a \in L \mid \delta^*(s_0, w) \in {}^aG = G(\delta^*(s_0, w))$ which immediately follows from the well-known relationship between an **L**-set (G, in our case) and the corresponding system ($\{{}^aG \mid a \in L\}$, in our case) of its a-cuts.

Conversely, let $\mathbf{S} = \{\mathscr{A}_a \mid a \in L\}$ be an **L**-nested system of deterministic automata $\mathscr{A}_a = \langle S, \Sigma, \delta, s_0, G_a\rangle$. Define the **L**-set G in S by $G(s) = \bigvee_{s \in G_a} a$. Now, since $G_a = {}^aG$, the required equality $(L(\mathbf{S}))(w) = (L(\mathscr{A}))(w)$ follows from the previous case. □

Remark.

(1) Note that the mappings defined in the proof of Theorem 2.2 sending a deterministic fuzzy automaton to an **L**-nested system of deterministic automata and conversely, are mutually inverse.
(2) In [2], it is proved that fuzzy automata are equally powerful as **L**-nested systems of non-deterministic automata (a special case for $L = [0, 1]$ is considered in [2]; however, this can be easily extended to arbitrary complete lattice). Theorems 2.1 and 2.2 thus imply that any two of the four acceptory devices (i.e. fuzzy automata, deterministic fuzzy automata, **L**-nested systems of non-deterministic automata, and **L**-nested systems of deterministic automata) are equally powerful.

3. Uncited reference

[3].

Acknowledgements

Support by NATO Advanced Fellowship B for 2000 is gratefully acknowledged.

References

[1] G.J. Klir, B. Yuan, Fuzzy Sets and Fuzzy Logic. Theory and Applications, Prentice-Hall, Englewood Cliffs, NJ, 1995.
[2] J. Močkoř, Fuzzy and non-deterministic automata, Soft Comput. 3 (1999) 221–226.
[3] V. Novák, Fuzzy Sets and Their Applications, Adam Hilger, Bristol, 1989.
[4] E.S. Santos, Maximin automata, Inf. Control 13 (1968) 363–377.
[5] W.G. Wee, K.S. Fu, A formulation of fuzzy automata and its application as a model of learning system, IEEE Trans. Syst. Sci. Cybern. 5 (3) (1969) 215–223.