7 Formal Concept Analysis in Geology: Attribute Tables, Their Analysis, and Discovery of Hidden Formal Concepts

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Abstract When exploring an unknown domain of interest, the primarily observable data are in the form of a collection of relevant objects (e.g. minerals, fossils) and their attributes (e.g. "is hard"). Analysis of data in the form of objects and their attributes is the primary aim of so-called Formal Concept Analysis. Recently, formal concept analysis has been developed also for situations where the object-attribute knowledge is vague (fuzzy), which is typical of sciences like geology, biology etc. Basically, formal concept analysis is able to help automatically answer the following questions: (1) What are the natural concepts that are hidden in the object-attribute data (e.g. important classes of organisms, minerals, fossils) and what is their natural hierarchical structure? (2) What are the attribute dependencies that are implicit in the object-attribute data?

The purpose of this chapter is to present formal concept analysis of fuzzy data and its possible applications in geological and related sciences. The presen-

tation is, however, written so as to be useful to experts from other fields as well. Presented notions and theorems are illustrated by examples. Main emphasis is put on providing the reader with an example-illustrated guide to methods of formal concept analysis so that he can apply it in his own domain of interest.

7.1 Introduction

Directly observable data: objects and their attributes. When humans formulate their knowledge about some domain of interest, they usually recognize *objects* and (their) *properties*. Objects and attributes (properties) are, indeed, primary phenomena when observing the physical world. When experts are going to explore an unknown area of interest, their first step is to identify relevant objects and their attributes. Then, experts have to identify what objects have which attributes. With this object-attribute knowledge at hand, experts then start further investigation. Typical examples of what is investigated are various kinds of relationships between attributes and a natural classification scheme. For instance, one looks for various dependencies among numerical attributes, for reducibility of attributes (a seemingly "elementary" attribute may be found to be a (hidden) combination of other attributes). The attributes can also be useful in devising suitable criteria according to which the objects relevant to the domain may be naturally classified. Furthermore, it is often found that, in order to get an insightful view into the domain of interest, one needs to establish a reasonable conceptual system, i.e. a collection of concepts (specific to the domain) with the basic relationships between the concepts.

Analysis: discovery of hidden attribute dependencies and natural concepts. What was outlined above is the more so true for biological sciences, geological sciences etc. Let us illustrate the above general description by an example. Suppose we arrive at a new territory with completely unknown living organisms (or suppose we are in our world but do not know anything about the organisms living here) and suppose we want to know more about the organisms. That means, we want to be able to do more than just to recognize and distinguish the organisms we encounter. The objects of our domain are thus the living organisms (or a suitable collection of them). We may now select several attributes of these organisms that seem to some extent relevant and elementary (directly observable). So far, our knowledge is limited to the knowledge of what objects have what attributes. Note that this knowledge is naturally depicted in the form of a (two dimensional) table with rows corresponding to objects and columns corresponding to attributes. Data entry corresponding to the table cell which is the intersection of x-th row (row corresponding to object x) and y-th column (column corresponding to attribute y) contains the value of attribute y on object x. This value may be a numerical value (if the attribute is e.g. the weight in kilograms), a logical value (in case of qualitative attributes like "to be hard"), or some other value (e.g. where the object was found). Typical of geological and biological sciences is the fact that some attributes are commonly fuzzy (e.g. "to be hard") in the sense that an attribute applies to an object only to a certain degree. To get a deeper insight into the object-attribute data, one naturally asks what are the dependencies and relationships among the attributes that can be read from the table. For instance, there might be dependencies, not visible on the surface, that tell us that a certain combination of certain attributes determine to some extent another attribute or attributes. As an example, consider a dependency "if x lives in the water then x has ..." One wishes to have an automatic procedure for obtaining the dependencies from the data. Another natural question relates to the fact that it is almost impossible to communicate knowledge without a conceptual system that is appropriate for a given domain. That is, one looks for what natural concepts are hidden in the data and what is the hierarchical structure of these concepts. For instance, one expects that some natural concepts like "a flying predator" are in some way hidden in the present object-attribute data, and that some natural hierarchy of those concepts is in the data as well.

7.2 Formal Concept Analysis: What and Why

Origins. The kind of data analysis outlined in the previous section is the main objective of formal concept analysis. The roots of formal concept analysis go to the paper by Wille [1982]. In this paper, he outlined his program of "restructuring lattice theory." The main aim was to develop lattice theory close to the original motivations of the theory of ordering. This is best illustrated by the following quotation from the paper:

> "The approach to lattice theory outlined in this paper is based on an attempt to reinvigorate the general view of order. For this purpose we go back to the origin of lattice concept in nineteenth-century attempts to formalize logic, where the study of hierarchies of concepts played a central role (cf. [Schröder 1890–95]). ... In set-theoretical language, this gives rise to lattices whose elements correspond to the concepts ... and whose order comes from the hierarchy of concepts."

The theory that resulted from this endeavour is called the theory of concept lattices. The part dealing with applications to analysis of object-attribute data is known as formal concept analysis. The basic reference is [Ganter and Wille, 1999] where one can also find an extensive list of publications related to both theory and applications. Extension of concept lattices and formal concept analysis to the case of fuzzy data (which is the tool we are interested in) can be found in [Bělohlávek, 2002], [Pollandt, 1997].

Informal outline. The rest of this section is devoted to an informal outline of the conceptual framework of formal concept analysis. The basic notion which serves to represent the input object-attribute knowledge is that of a formal context. Formal context consists of a set X of objects, a set Y of attributes, and a relation I between objects and attributes. The set X represents objects that are relevant to our domain of interest, i.e. objects to which we restrict our attention. Likewise, Y contains relevant attributes. We restrict ourselves to the case where attributes are qualitative, i.e. they either apply or don't apply or apply only to a certain truth degree. Examples of such attributes are "to be

	y_1	$\dots$	y_j	$\dots$	y_l
x_1			$\vdots$		
$\vdots$			$\vdots$		
x_i	$\dots$	$\dots$	$I(x_i, y_j)$	$\dots$	$\dots$
$\vdots$			$\vdots$		
x_k			$\vdots$		

Table 7.1: Input data in the tabular form. $I(x_i, y_j)$ is the degree to which attribute y_j applies to object x_i.

found in North America" (provided North America is sharply delineated, this is an example of so-called crisp attribute; each object x either was found in North America (the attribute takes logical value 1 on this object) or was not (the attribute takes logical value 0 on this object)) or "to be hard" (this is an example of a typical fuzzy attribute; the fact that an object is hard may be assigned a truth degree, say, 0.7 if the object is more or less hard). Therefore, the input data specify for each object x from X and each attribute y from Y to which extent the attribute y applies to the object x. This is naturally done by specifying the truth degree $I(x, y)$ of the fact "y applies to x." The degree $I(x, y)$ is supplied by an expert's observation of the domain. In practice, the sets X and Y, are of course finite. The input data can thus be put into a table specifying the values $I(x, y)$ for each x from X and y from Y. Let the elements of X be $x_1, \dots, x_k$, and the elements of Y be $y_1, \dots, y_l$. Then the input data, i.e. the triple $\langle X, Y, I\rangle$ can be represented by a table shown in Table 7.1.

Having specified the input data $\langle X, Y, I\rangle$, one is interested in the analysis of this data. In this preliminary outline, we focus on the discovery of natural concepts hidden in the data, the discovery of attribute dependencies in the form of attribute implications, and the measuring of similarity of attributes and similarity of the discovered concepts.

Discovering natural concepts hidden in the data. First of all, one needs to say what is to be understood by a (formal) concept. In formal concept analysis, a concept is understood according to a long-standing tradition of Port-Royal logic [Arnauld and Nicole, 1662]. A concept is determined by its *extent* and its *intent*. The extent of a concept is the collection of all objects that are covered by the concept. The intent of a concept is the collection of all attributes covered by the concept. For instance, consider the concept DOG. The extent of DOG consists of all dogs while the intent of DOG consists of all attributes that apply to dogs (like "to be a mammal", "to bark" etc.). Therefore, a concept in formal concept analysis is understood to be a pair $\langle A, B\rangle$ consisting of a collection A of objects (extent) and a collection B of attributes (intent). In order to qualify for a concept, the pair $\langle A, B\rangle$ has to satisfy some constraints. Note that this is extremely important and makes formal concept analysis what it is. If there were no constraint, the whole thing would be useless; in this case, each pair $\langle A, B\rangle$ would be a concept and so "being a concept" would carry no information. The constraint being used in formal concept analysis is very simple and can be described verbally as follows: A pair $\langle A, B\rangle$ is called a (formal) concept if

> B is the collection of all attributes shared by all objects from A

and

> A is the collection of all objects sharing all the attributes from B.

Therefore, given the input data $\langle X, Y, I\rangle$, there might be several pairs $\langle A, B\rangle$ where A is a subcollection of X and B is a subcollection of Y that satisfy the definition of a formal concept. These formal concepts are *hidden* in the input data in that their presence is not obvious by just looking at the table. A procedure that takes $\langle X, Y, I\rangle$ as its input and generates the list of formal concepts that are hidden in $\langle X, Y, I\rangle$ may be considered as performing a *discovery of natural concepts* that "exist in the data." Formal concepts that arise from such a procedure are, in a sense, meaningful clusters of objects and attributes (where

	y_1	y_2	y_3	y_4
x_1	1	1	0	0
x_2	0	1	1	0
x_3	0	0	1	1

Table 7.2: Input data to Example 7.1.

'meaningful' is to be considered with respect to (w.r.t.) the conceptual interpretation). We denote the collection of all formal concepts hidden in $\langle X, Y, I\rangle$ by $\mathcal{B}(X, Y, I)$. Let us illustrate the notion of a concept by a simple example.

Example 7.1 We present a simple example illustrating the notions. Let $X = \{x_1, x_2, x_3\}$, $Y = \{y_1, y_2, y_3, y_4\}$ and consider a binary relation I given by Table 7.2. That is, x_1 has attributes y_1 and y_2 but does not have attributes y_3 and y_4. Although this is a very simple example, we will interpret x_i and y_j as follows. Let x_1, x_2, x_3 be some geological objects, say minerals; let y_1 mean "was found in North America", y_2 mean "was found in South America", y_3 mean "was found in Asia", y_3 mean "was found in Europe". However, note that "object" is just a technical term. In our example, x_i refers to a whole group of minerals of the same sort (one particular mineral cannot be found in both North America and South America). There are seven concepts hidden in this data; they are listed in Table 7.3. For instance, concept c_4 is a pair $\langle A_4, B_4\rangle$ with the extent $A_4 = \{x_1, x_2\}$ and the intent $B_4 = \{y_2\}$. That is c_4 covers mineral x_1 and mineral x_2, and covers attribute "found in South America". We comment more on the concepts below.

Hierarchy of discovered concepts. The next issue we go into is the hierarchy of discovered concepts. Hierarchy of concepts w.r.t. their generality is the basic relation that accompanies concepts. For instance, the concept DOG is a subconcept of the concept MAMMAL, MAMMAL is a superconcept of DOG. We denote the conceptual hierarchy by $\leq$ and write $\langle A_1, B_1\rangle \leq \langle A_2, B_2\rangle$ to denote that the concept $\langle A_1, B_1\rangle$ is a subconcept of the concept $\langle A_2, B_2\rangle$. Being more

	x_1	x_2	x_3	y_1	y_2	y_3	y_4
c_1	1	1	1	0	0	0	0
c_2	0	1	1	0	0	1	0
c_3	0	0	1	0	0	1	1
c_4	1	1	0	0	1	0	0
c_5	0	1	0	0	1	1	0
c_6	1	0	0	1	1	0	0
c_7	0	0	0	1	1	1	1

Table 7.3: Concepts from Example 7.1.

general means to cover more objects (or, which is equivalent, less attributes). Therefore, it is only natural to define $\leq$ by

> $\langle A_1, B_1\rangle \leq \langle A_2, B_2\rangle$ if and only if A_1 is a subcollection of A_2 or, equivalently, if and only if B_2 is a subcollection of B_1.

It is easily seen that the thus introduced hierarchy is a partial order, i.e. it is reflexive ($c \leq c$), antisymmetric ($c_1 \leq c_2$ and $c_2 \leq c_1$ imply $c_1 = c_2$) and transitive ($c_1 \leq c_2$ and $c_2 \leq c_3$ imply $c_1 \leq c_3$). Moreover, for each collection of formal concepts from $\mathcal{B}(X, Y, I)$ there exits both their direct superconcept (generalization) and their direct subconcept (specialization) in $\mathcal{B}(X, Y, I)$ (see the next section). Therefore, $\leq$ obeys the laws naturally required for a complete conceptual system. The hierarchical structure of the collection $\mathcal{B}(X, Y, I)$ w.r.t. the hierarchy order $\leq$ is easily depicted by a diagram (so-called Hasse diagram). The next example serves to illustrate this.

Example 7.2 Consider the input data from Example 7.1. The hierarchical structure of $\mathcal{B}(X, Y, I)$ is depicted in Figure 7.1. Concept c_7 is the most general concept; its extent contains all objects (x_1, x_2, x_3). On the other hand, c_1 is the empty concept; its extent does not contain any object. Between c_1 and c_7 the are five concepts. For instance, concepts c_3 and c_5 have the verbal descriptions "to be found in Asia and in Europe" and "to be found in South America and in

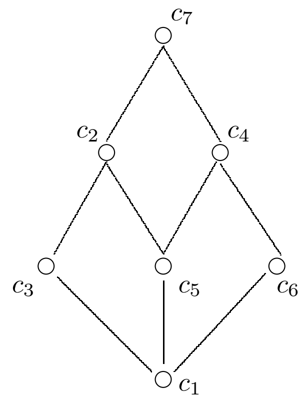

Figure 7.1: Hierarchy of hidden formal concepts from Example 7.1.

Asia", respectively. Concept c_2 is the join of c_3 and c_5 and therefore, c_2 is the direct generalization of c_3 and c_5. Indeed, the intent of c_2 is $\{y_3\}$ which means that the verbal description of c_2 is "to be found in Asia".

Attribute dependencies. Attribute dependencies, as approached in formal concept analysis, are expressed by implications of the form

attributes $y_1, \ldots z_1$ imply attributes $y_2, \ldots, z_2$,

written formally $\{y_1, \ldots, z_1\} \Rightarrow \{y_2, \ldots, z_2\}$. The meaning of the above implication is that each object that has all of the attributes $y_1, \ldots, z_1$ has also all of the attributes $y_2, \ldots, z_2$; it is this sense in which the implication is valid in a given input data $\langle X, Y, I \rangle$. The attribute implications discover other kinds of knowledge hidden in the input data. The analysis of attributes may reveal that some attributes are only simple combinations of others. We are surely interested in all implications that are valid in the input data. However, listing of all implications can yield a huge number of implications. Moreover, some of them are trivial (like $\{y_1\} \Rightarrow \{y_1\}$), some of them follow (in a natural but precisely defined sense) from other implications. Formal concept analysis has means to generate only basic implications (in the sense that all other implications follow from the basic ones). The next example illustrates the basics of attribute implications.

Example 7.3 Consider again the input data of Example 7.1. We can see that there are, for instance the attribute implications $\{y_4\} \Rightarrow \{y_3\}$ and $\{y_1\} \Rightarrow \{y_2\}$ which are true in the input data. $\{y_4\} \Rightarrow \{y_3\}$ says that each mineral found in Europe was also found in Asia and $\{y_1\} \Rightarrow \{y_2\}$ says that each mineral found in North America was also found in South America. On the contrary, implication $\{y_3\} \Rightarrow \{y_4\}$ is not true in the data since x_3 has the attribute y_3 but does not have the attribute y_4.

Fuzziness and similarity issues. The example we used to demonstrate the basic notions of formal concept analysis was specific in that the attributes were crisp. However, most empirical attributes are fuzzy. What was described above also applies if attributes are fuzzy, we only need correctly interpret the verbal description (see the next section). In case the input data contains fuzzy attributes, an important phenomenon is that of similarity. Similarity can be basically considered on three levels: similarity of attributes (and similarity of objects); similarity of concepts; and similarity of the conceptual structures.

Similarity is a graded notion. Objects x and y may be more similar than objects x and z are. Simplifying reality using similarity is the very nature of how humans cope with the complexity of outer world. Basically, the simplification is done by identifying objects that are "very similar." The process of identification of elements is known as a factorization. It is the usual formal model of what people call abstraction. What "very similar" means depends on how coarse the factorization (how much abstraction) is required. Fuzzy logic has natural means to model factorization w.r.t. graded similarity.

Intuitively, we consider two attributes similar if they apply to each object of the domain of discourse approximately to the same extent. This makes it possible to reduce the input data, i.e. to identify attributes that are "very much" similar. Distinguishing very similar attributes would lead to overly detailed and extensive analysis.

From the input (fuzzy) data one can generate the list of all formal concepts hidden in the data and the hierarchical structure of the concepts. A natural question is that of how the similarity of the formal concepts can be measured. Intuitively, we consider two concepts similar if they apply to all objects in approximately the same extent. If one finds one does not need that level of discernibility which is represented by the generated structure of concepts, one may wish to simplify the conceptual system by identifying concepts that are sufficiently similar. Doing so, one obtains a simpler conceptual system that is (for the given purposes) sufficient.

Finally, a natural problem is the similarity of two conceptual systems. A conceptual system is a system of basic abstract units (concepts) which allows efficient communication. Given two systems, an immediate question is to what extent are the two equivalent in that each concept of one of them may be described by concepts in the other one.

Formal concept analysis of fuzzy data has means for naturally modeling all of the three above described levels of similarity. Basically, it answers the questions of (a) how to measure similarity, and (b) how to simplify (factorize) by similarity.

7.3 Formal Concept Analysis of Fuzzy Data: a Guided Tour

This section presents basic notions and results of formal concep analysis of fuzzy data. Theorems are presented without proofs which can be found in [Bělohlávek, 2002b].

Fuzzy Context and Fuzzy Concepts: Input Data and Hidden Concepts

Let X be a set of objects and Y be a set of attributes to which we restrict our attention. Let L be a (suitable) set of truth degrees. Furthermore, let I be a binary fuzzy relation with truth degrees in L; that is, I assigns to each $x \in X$ and each $y \in Y$ a truth degree $I(x, y) \in L$. The degree $I(x, y)$ is interpreted as the truth degree to which object x has attribute y.

Definition 7.4 The above triple $\langle X, Y, I \rangle$ is called a *fuzzy context.*

Mostly, one takes L to be some subset of $[0,1]$. As we will see, we need operations on L that correspond to logical connectives. That is, L shall be equipped with a couple of operations corresponding to conjunction, implication etc. We will provide a general structure for this purpose (L will be equipped with a structure of so-called complete residuated lattice) and then show particular examples of this structure that are most commonly used in applications.

Complete residuated lattices are the basic structures of truth values used in fuzzy logic in narrow sense [Höhle, 1996], [Hájek, 1998]. The reader can find basic information about lattices and partially ordered sets in [Davey and Priestley, 1990]. A complete residuated lattice is an algebra $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ such that $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice with the least element 0 and the greatest element 1; $\langle L, \otimes, 1\rangle$ is a commutative monoid, i.e. $\otimes$ is commutative, associative, and $x \otimes 1 = x$ holds for each $x \in L$; and $\otimes, \rightarrow$ are binary operations which form an adjoint pair, i.e. $x \otimes y \leq z$ if and only if $x \leq y \rightarrow z$ holds for all $x, y, z \in L$. The operation $\otimes$ corresponds to conjunction, $\rightarrow$ corresponds to implication.

The most studied and applied set of truth values is the real interval $[0,1]$. Each left-continuous t-norm $\otimes$ induces a complete residuated lattice $\langle [0,1], \min, \max, \otimes, \rightarrow, 0, 1\rangle$ where $\rightarrow$ is given by $a \rightarrow b = \max\{c \mid a \otimes c \leq b\}$ (and conversely, each residuated lattice on $[0,1]$ is induced in this way by some left-continuous t-norm), for details and more information see [Bělohlávek, 2002b], [Hájek, 1998]. The most popular t-norms are: the Łukasiewicz t-norm ($a \otimes b = \max(a + b - 1, 0)$, $a \rightarrow b = \min(1 - a + b, 1)$), Gödel t-norm ($a \otimes b = \min(a, b)$, $a \rightarrow b = 1$ if $a \leq b$ and $a \rightarrow b = b$ else), and product t-norm ($a \otimes b = a \cdot b$, $a \rightarrow b = 1$ if $a \leq b$ and $a \rightarrow b = b/a$ else). They are in fact continuous. On the other hand, any continuous t-norm can be composed in a simple way out of these three, see [Bělohlávek, 2002b], [Hájek, 1998]. Another important set of truth values is the set $\{a_0 = 0, a_1, \ldots, a_n = 1\}$, $a_0 < \cdots < a_n$, where the ordering determines the complete lattice structure. Two t-norms are often considered: $a_k \otimes a_l = a_{\max(k+l-n,0)}$ and the corresponding $\rightarrow$ given by $a_k \rightarrow a_l = a_{\min(n-k+l,n)}$ (Łukasiewicz) and $a_k \otimes a_l = a_{\min(k,l)}$ and the corresponding $\rightarrow$ given by $a_k \rightarrow a_l = 1$ if $k \leq l$ and a_l else (Gödel). A special case of the latter algebras is the Boolean algebra **2** of classical logic with the support set $\{0, 1\}$. It may be easily verified that the only t-norm on $\{0, 1\}$ is the classical conjunction operation $\wedge$, i.e. $a \wedge b = 1$ if and only if $a = 1$ and $b = 1$, which implies that the only residuum operation is the classical implication operation $\rightarrow$, i.e. $a \rightarrow b = 0$ if and only if $a = 1$ and $b = 0$. Note that each of the preceding residuated lattices is complete.

In the following, $\mathbf{L}$ always denotes some complete residuated lattice. However, there will be no substantial lost if one assumes that L is $[0, 1]$ or some finite subchain of $[0, 1]$.

A *fuzzy set* (or $\mathbf{L}$-set) A in a universe set X is a mapping $A : X \rightarrow L$. The value $A(x) \in L$ is interpreted as the truth value of the statement "the element x belongs to A". The set of all fuzzy sets in X is denoted by L^X. For $A_1, A_2 \in L^X$ we write $A_1 \subseteq A_2$ if and only if $A_1(x) \leq A_2(x)$ for all $x \in X$. Similarly, a binary fuzzy relation R between X and Y is a mapping $R : X \times Y \rightarrow L$. Particularly, $\{\, {}^{a_1}\!/x_1, \cdots, {}^{a_n}\!/x_n\}$ denotes a fuzzy set A with $A(x_1) = a_1, \ldots, A(x_n) = a_n$, and $A(x) = 0$ for $x \neq x_i$ $(i = 1, \ldots, n)$.

Coming back to the notion of a fuzzy context, we want to formalize the notion of a concept. According to Port-Royal, a concept consists of a collection of objects (its extent) and a collection of attributes (its intent). If the attributes in the context are fuzzy, both extent and intent are assumed to be fuzzy sets. Consider e.g. the concept hard mineral. Clearly, its extent is a fuzzy set rather than a crisp set. Following the verbal definition, we need to define two operators, $^\uparrow$ and $^\downarrow$. The intended meaning of $^\uparrow$ and $^\downarrow$ is the following. For a fuzzy set A of objects (i.e. $A \in L^X$), $A^\uparrow$ is the fuzzy set of all attributes (i.e. $A^\uparrow \in L^Y$) shared by all objects from A; for a fuzzy set B of attributes (i.e. $B \in L^Y$), $B^\downarrow$ is the fuzzy set of all objects (i.e. $B^\downarrow \in L^X$) sharing all attributes from B. The basic semantical rules of fuzzy logic give the following. For a fuzzy context $\langle X, Y, I\rangle$, $A \in L^X$, and $B \in L^Y$, $A^\uparrow$ and $B^\downarrow$ are a fuzzy set in Y and a fuzzy set in X, respectively, defined by

$$A^{\uparrow_I}(y) = \bigwedge_{x \in X} A(x) \to I(x,y) \tag{7.1}$$

$$B^{\downarrow_I}(x) = \bigwedge_{y \in Y} B(y) \to I(x,y). \tag{7.2}$$

for each $y \in Y$ and $x \in X$. Therefore, (7.1) and (7.2) define mappings ${}^{\uparrow_I}$: $L^X \to L^Y$, ${}^{\downarrow_I} : L^Y \to L^X$. If I is obvious, we write only ${}^{\uparrow}$ and ${}^{\downarrow}$ instead of ${}^{\uparrow_I}$ and ${}^{\downarrow_I}$.

Example 7.5 Let $X = \{x_1, x_2, x_3\}$, $Y = \{y_1, y_2\}$, $L = [0,1]$, $I(x_1, y_1) = 1$, $I(x_1, y_2) = 0.3$, $I(x_2, y_1) = 0.8$, $I(x_2, y_2) = 0.9$, $I(x_3, y_1) = 0$, $I(x_3, y_2) = 0.1$. Consider $A = \{ {}^{0.5}/x_1,\ {}^{1}/x_2,\ {}^{0}/x_3\}$, $B = \{ {}^{0.7}/y_1,\ {}^{0.3}/y_2\}$. We want to determine $A^{\uparrow_I}$ and $B^{\downarrow_I}$. For instance, with the Łukasiewicz structure on $[0,1]$, we have

$$\begin{aligned} & A^{\uparrow_I}(y_1) = \bigwedge_{x \in X} A(x) \to I(x, y_1) \\ = \; & [A(x_1) \to I(x_1, y_1)] \wedge [A(x_2) \to I(x_2, y_1)] \wedge [A(x_3) \to I(x_3, y_1)] \\ = \; & [0.5 \to 1] \wedge [1 \to 0.8] \wedge [0 \to 0] \\ = \; & 1 \wedge 0.8 \wedge 1 = 0.8, \end{aligned}$$

$$\begin{aligned} & A^{\uparrow_I}(y_2) = \bigwedge_{x \in X} A(x) \to I(x, y_2) \\ = \; & [A(x_1) \to I(x_1, y_2)] \wedge [A(x_2) \to I(x_2, y_2)] \wedge [A(x_3) \to I(x_3, y_2)] \\ = \; & [0.5 \to 0.3] \wedge [1 \to 0.9] \wedge [0 \to 0.1] \\ = \; & 0.8 \wedge 0.9 \wedge 1 = 0.8, \end{aligned}$$

$$\begin{aligned} & B^{\downarrow_I}(x_1) = \bigwedge_{y \in Y} B(y) \to I(x_1, y) \\ = \; & [B(y_1) \to I(x_1, y_1)] \wedge [B(y_2) \to I(x_1, y_2)] \\ = \; & [0.7 \to 1] \wedge [0.3 \to 0.3] \\ = \; & 1 \wedge 1 = 1, \end{aligned}$$

$$\begin{aligned} B^{\downarrow_I}(x_2) &= \bigwedge_{y\in Y} B(y) \to I(x_2,y) \\ &= [B(y_1) \to I(x_2,y_1)] \wedge [B(y_2) \to I(x_2,y_2)] \\ &= [0.7 \to 0.8] \wedge [0.3 \to 0.7] \\ &= 1 \wedge 1 = 1, \end{aligned}$$

$$\begin{aligned} B^{\downarrow_I}(x_3) &= \bigwedge_{y\in Y} B(y) \to I(x_3,y) \\ &= [B(y_1) \to I(x_3,y_1)] \wedge [B(y_2) \to I(x_3,y_2)] \\ &= [0.7 \to 0] \wedge [0.3 \to 0.1] \\ &= 0.3 \wedge 0.8 = 0.3. \end{aligned}$$

One may check that changing the structure on $[0,1]$ changes the operators $^{\uparrow_I}$ and $^{\downarrow_I}$. For instance, with the Gödel structure on $[0,1]$, we get $A^{\uparrow_I}(y_1) = 0.8$, $A^{\uparrow_I}(y_2) = 0.3$, $B^{\downarrow_I}(x_1) = 1$, $B^{\downarrow_I}(x_2) = 1$, $B^{\downarrow_I}(x_3) = 0$.

Then the verbal definition of a concept may be formalized as follows:

Definition 7.6 A *fuzzy concept* in a fuzzy context $\langle X, Y, I\rangle$ is each pair $\langle A, B\rangle$ of a fuzzy set $A \in L^X$ of objects and a fuzzy set $B \in L^Y$ of attributes such that $A^{\uparrow} = B$ and $B^{\downarrow} = A$.

Indeed, this definition just makes formal the verbal constraints that have to be fulfilled by the extent A and the intent B of a concept consisting of A and B.

The set of all fuzzy concepts in a given fuzzy context $\langle X, Y, I\rangle$ will be denoted by $\mathcal{B}(X, Y, I)$. That is, we have

$$\mathcal{B}(X, Y, I) = \{\langle A, B\rangle \in L^X \times L^Y \mid A^{\uparrow_I} = B, B^{\downarrow_I} = A\}.$$

$\mathcal{B}(X, Y, I)$ will be called a *fuzzy concept lattice* determined by a fuzzy context $\langle X, Y, I\rangle$ (the term concept "lattice" will be justified later). We write only "context", "concept" and "concept lattice" if $\mathbf{L}$ is obvious. On the other hand, if we want to emphasize the structure $\mathbf{L}$ of truth values, we write "$\mathbf{L}$-context", "$\mathbf{L}$-concept" and "$\mathbf{L}$-concept lattice". It is important to note that if $\mathbf{L} = \mathbf{2}$, i.e. the structure of truth values is the two-element set $L = \{0, 1\}$, then the notions of an $\mathbf{L}$-context, $\mathbf{L}$-concept and $\mathbf{L}$-concept lattice become the notions of a (crisp) context, (crisp) concept, and a (crisp) fuzzy concept lattice developed by (Wille, 1982).

Example 7.7 For $\mathbf{L} = \mathbf{2}$ (i.e. $L = \{0, 1\}$, thus we have 0 and 1 as the only truth values), we have that $A^{\uparrow}(y) = 1$ (i.e. y belongs to $A^{\uparrow}$) if and only if for each $x \in X$ such that $A(x) = 1$ we have $I(x, y) = 1$, or, in other words, each $x \in A$ is in relation I with y, which is the meaning of $A^{\uparrow}$ in the crisp case. The situation for $B^{\downarrow}$ is symmetric.

Fuzzy concepts may be viewed as maximal rectangles in the object-attribute table corresponding to a fuzzy context. Although this is especially appealing in the crisp case, i.e. $\mathbf{L} = \mathbf{2}$, we will demonstrate this alternative view of concepts in general. A binary fuzzy relation I' between X and Y is called a *rectangular relation* if and only if there are $A \in L^X$, $B \in L^Y$ such that $I'(x, y) = A(x) \otimes B(y)$ (for all $x \in X$, $y \in Y$), written $I' = A \otimes B$. In this case, the pair $\langle A, B\rangle$ is called a *rectangle*. A rectangle $\langle A, B\rangle$ is said to be *contained* in a binary fuzzy relation I'' if $A \otimes B$ is contained in I'', i.e. if $A \otimes B \subseteq I''$. There is a naturally defined ordering $\leq$ defined on the set of all rectangles by $\langle A_1, B_1\rangle \leq \langle A_2, B_2\rangle$ if and only if for all $x \in X$, $y \in Y$ we have $A_1(x) \leq A_2(x)$ and $B_1(y) \leq B_2(y)$. The following theorem, if interpreted for $\mathbf{L} = \mathbf{2}$, says that concepts are just maximal rectangles of I which are filled with 1's (if we consider the two-valued relation I as a matrix-table of 0's and 1's).

Theorem 7.8 *For a fuzzy context $\langle X, Y, I\rangle$ and $A \in L^X$, $B \in L^Y$, we have that $\langle A, B\rangle$ is a fuzzy concept iff it is a maximal rectangle contained in I.*

Fuzzy Concept Lattices: Hierarchy of Hidden Concepts

We are now going to investigate the set of all fuzzy concepts with its hierarchical structure. A thorough treatment on this topic can be found in (Bělohlávek, to appear). At this moment, we confine ourselves to a special case: we concentrate on the "crisp" hierarchy of fuzzy concepts. The subconcept-superconcept relation $\leq$ on the set $\mathcal{B}(X, Y, I)$ of all fuzzy concepts in a fuzzy context $\langle X, Y, I\rangle$ (i.e. the conceptual hierarchy) is naturally defined by

$$\langle A_1, B_1\rangle \leq \langle A_2, B_2\rangle \quad \text{iff} \quad A_1 \subseteq A_2 \quad (\text{ iff } \ B_1 \supseteq B_2). \tag{7.3}$$

That is, the fuzzy concept $\langle A_1, B_1\rangle$ is a subconcept of the fuzzy concept $\langle A_2, B_2\rangle$ if the extent A_2 is greater than the extent A_1. This means that the degree to which any object x belongs to A_2 is at least as high as the degree to which x belongs to A_1. Equivalently, $\langle A_1, B_1\rangle$ is a subconcept of $\langle A_2, B_2\rangle$ if the intent B_1 is greater than the intent B_2. Saying that $\langle A_1, B_1\rangle$ is a subconcept of $\langle A_2, B_2\rangle$ may be equivalently expressed by saying that $\langle A_1, B_1\rangle$ is more special than $\langle A_2, B_2\rangle$, or that $\langle A_2, B_2\rangle$ is more general than $\langle A_1, B_1\rangle$, or that $\langle A_2, B_2\rangle$ is a superconcept of $\langle A_1, B_1\rangle$.

The structure of $\mathcal{B}(X, Y, I)$ w.r.t. $\leq$ and its characterization is the subject of the following theorem.

Theorem 7.9 (main theorem of fuzzy concept lattices, crisp order) *Let $\langle X, Y, I\rangle$ be an $\mathbf{L}$-context. (1) $\mathcal{B}(X, Y, I)$ is a complete lattice in which infima and suprema can be described as follows:*

$$\bigwedge_{j\in J} \langle A_j, B_j\rangle = \left\langle \bigcap_{j\in J} A_j, (\bigcap_{j\in J} A_j)^{\uparrow} \right\rangle = \left\langle \bigcap_{j\in J} A_j, (\bigcup_{j\in J} B_j)^{\downarrow\uparrow} \right\rangle, \tag{7.4}$$

$$\bigvee_{j\in J} \langle A_j, B_j\rangle = \left\langle (\bigcap_{j\in J} B_j)^{\downarrow}, \bigcap_{j\in J} B_j \right\rangle = \left\langle (\bigcup_{j\in J} A_j)^{\uparrow\downarrow}, \bigcap_{j\in J} B_j \right\rangle. \tag{7.5}$$

(2) Moreover, a complete lattice $\mathbf{V} = \langle V, \leq\rangle$ is isomorphic to $\mathcal{B}(X, Y, I)$ iff there are mappings $\gamma : X \times L \to V$, $\mu : Y \times L \to V$, such that $\gamma(X \times L)$ is supremally dense in $\mathbf{V}$, $\mu(Y \times L)$ is infimally dense in $\mathbf{V}$, and $a \otimes b \leq I(x, y)$ is equivalent to $\gamma(x, a) \leq \mu(y, b)$ for all $x \in X$, $y \in Y$, $a, b \in L$.

Recall that the fact that $\mathcal{B}(X, Y, I)$ is a complete lattice is a very natural one. Recall (Birkhoff, 1967) that a complete lattice is a partially ordered set V where to each subset V' of V there exists the infimum of V' as well as the supremum. Now the supremum of V' is the least one of all elements of V which are greater than each element of V'. That is, under the conceptual interpretation, if V' is a set of concepts then the supremum of V' is a concept which can be thought of as the direct generalization of all concepts from V'. Dually, the infimum of V' is the concept which can be thought of as the direct common specialization of the concepts from V'.

Attribute Implications

Attribute implications represent another useful set of information that can be extracted from the input data (fuzzy context). The basic idea is this: An *attribute implication* consists of a pair consisting of a fuzzy set A of attributes and a fuzzy set B of attributes. Such an attribute implication will be briefly denoted by $A \Rightarrow B$ (note that we use $\rightarrow$ to denote residuum (implication operation) in structures of truth values and also to denote attribute implications—however, there is no danger of confusion). Now, given the input data, i.e. a fuzzy context $\langle X, Y, I\rangle$, an attribute implication may be true in $\langle X, Y, I\rangle$ to a certain degree. The degree $A \Rightarrow_{\langle X,Y,I\rangle} B$ to which $A \Rightarrow B$ is true in $\langle X, Y, I\rangle$ is verbally described as the degree to which "for each $x \in X$: if x has all the attributes from A then x has also all the attributes from B". Formally, we have

$$A \Rightarrow_{\langle X,Y,I\rangle} B \;=\; \bigwedge_{x\in X} (\bigwedge_{y\in Y} (A(y) \rightarrow I(x, y)) \rightarrow \bigwedge_{y\in Y} (B(y) \rightarrow I(x, y))).$$

In a similar way, one can consider the degree to which an attribute implication is true in $\mathcal{B}(X, Y, I)$. We first elaborate on the details. Then we go to the notion of entailment of attribute implications and to the notion of a base of attribute implications. Recall that for fuzzy sets $C, D \in L^U$, the degree $S(C, D)$ to which C is contained in D is defined by $S(C, D) = \bigwedge_{x\in U}(C(x) \rightarrow D(x))$.

Definition 7.10 For a fuzzy context $\langle X, Y, I\rangle$, a fuzzy set $M \in L^Y$, and an attribute implication $A \Rightarrow B$, we put

$$\models (M, A \Rightarrow B) = S(A, M) \rightarrow S(B, M)$$

and call $\models (M, A \Rightarrow B)$ the *degree to which $A \Rightarrow B$ is true in M*.

Definition 7.10 has a natural interpretation: $\models (M, A \Rightarrow B)$ is the truth degree to which it is true that whenever A is contained in M then B is as well.

We are going to define the notion of validity of a collection of attribute implications in a collection of fuzzy sets of attributes. $\models$ is an **L**-relation between the set L^Y of all **L**-sets of attributes and the set $L^Y \times L^Y$ of all **L**-attribute implications. As established in [Bělohlávek, 1999], $\models$ induces an **L**-Galois connection $\langle ^\wedge, ^\vee\rangle$ between L^Y and $L^Y \times L^Y$, i.e. for $\mathcal{M} \in L^{L^Y}$ and $\mathcal{I} \in L^{L^Y \times L^Y}$ the **L**-sets $\mathcal{M}^\wedge \in L^{L^Y \times L^Y}$ and $\mathcal{I}^\vee \in L^{L^Y}$ given by

$$\begin{aligned} \mathcal{M}^\wedge(A \Rightarrow B) &= \bigwedge_{M \in L^{L^Y}} \mathcal{M}(M) \rightarrow \models (M, A \Rightarrow B) \\ \mathcal{I}^\vee(M) &= \bigwedge_{(A \Rightarrow B) \in L^{L^Y \times L^Y}} \mathcal{I}(A \Rightarrow B) \rightarrow \models (M, A \Rightarrow B). \end{aligned}$$

Therefore, $\mathcal{M}^\wedge(A \Rightarrow B)$ is the truth degree to which $A \Rightarrow B$ is true in each M from $\mathcal{M}$, and $\mathcal{I}^\vee(M)$ is the truth degree to which each implication from $\mathcal{I}$ is true in M.

Particularly, we will be interested in $\{\{\,^1\!/x\}^\uparrow \mid x \in X\}^\wedge$ and $\mathrm{Int}(X, Y, I)^\wedge$: $\{\{\,^1\!/x\}^\uparrow \mid x \in X\}^\wedge(A \Rightarrow B)$ is the truth degree to which $A \Rightarrow B$ is true in each $\{\,^1\!/x\}^\uparrow$ (i.e. the intent of the elementary fuzzy concept $\langle\{\,^1\!/x\}^{\uparrow\downarrow}, \{\,^1\!/x\}^\uparrow\rangle$) and $\mathrm{Int}(X, Y, I)^\wedge(A \Rightarrow B)$ is the truth degree to which $A \Rightarrow B$ is true in all intents of $\mathcal{B}\,(X, Y, I)$. For the sake of brevity we denote for $\mathcal{M} \in L^{L^Y}$ and $A \Rightarrow B$ the degree $\mathcal{M}^\wedge(A \Rightarrow B)$ by $A \Rightarrow_{\mathcal{M}} B$ and, furthermore, if $\mathcal{M} = \{\{\,^1\!/x\}^\uparrow \mid x \in X\}$ we write $A \Rightarrow_{\langle X,Y,I\rangle} B$ instead of $A \Rightarrow_{\mathcal{M}} B$. This makes a good sense: since $\{\,^1\!/x\}^\uparrow(y) = I(x, y)$, $A \Rightarrow_{\langle X,Y,I\rangle} B$ is the degree to which it is true that for each $x \in X$, if x has all the attributes from A then x has all the attributes of B; i.e. we get the notion of a validity of an attribute implication in a fuzzy context. For

$M = \langle D \mid \langle C, D\rangle \in \mathcal{B}(X, Y, I)\rangle$ we denote $A \Rightarrow_M B$ by $A \Rightarrow_{\mathcal{B}(X,Y,I)} B$. That is, $A \Rightarrow_{\mathcal{B}(X,Y,I)} B$ is the degree to which it is true that if A is contained in an intent D of some fuzzy concept $\langle C, D\rangle$ from the fuzzy concept lattice $\mathcal{B}(X, Y, I)$, then B is contained in B as well.

Theorem 7.11 *For any fuzzy context $\langle X, Y, I\rangle$ and any attribute implication $A \Rightarrow B$ we have*

$$A \Rightarrow_{\langle X,Y,I\rangle} B \quad = \quad A \Rightarrow_{\mathcal{B}(X,Y,I)} B,$$

i.e. for any implication $A \Rightarrow B$, the degree to which $A \Rightarrow B$ is valid in the fuzzy context $\langle X, Y, I\rangle$ equals the degree to which $A \Rightarrow B$ is true in $\mathcal{B}(X, Y, I)$.

Since $A \Rightarrow_{\langle X,Y,I\rangle} B$ and $A \Rightarrow_{\mathcal{B}(X,Y,I)} B$ coincide, we denote both of them simply by $A \Rightarrow_I B$ or even by $A \Rightarrow B$. The following assertions list some basic rules of the calculus of attribute implications.

Theorem 7.12 *For any fuzzy context $\langle X, Y, I\rangle$ and any attribute implication $A \Rightarrow B$, the truth degrees $A \Rightarrow B$, $S(A^{\downarrow}, B^{\downarrow})$, and $S(B, A^{\downarrow\uparrow})$ are equal.*

Theorem 7.13 *For each fuzzy context $\langle X, Y, I\rangle$ we have*

$$A \Rightarrow_I A = 1, \qquad (A \Rightarrow_I B) \otimes (B \Rightarrow_I C) \leq (A \Rightarrow_I C) \tag{7.6}$$

$$S(A_1, A_2) \otimes S(B_2, B_1) \otimes (A_1 \Rightarrow B_1) \leq (A_2 \Rightarrow B_2). \tag{7.7}$$

Thus, (7.6) says that A always implies A, and that if A implies B and B implies C then A implies C; (7.7) says that if A_1 implies B_1 and if A_1 is contained in A_2 and B_1 contains B_2, then A_2 implies B_2.

These rules indicate that some of the attribute implications "follow" from other implications. Therefore, it is not necessary to list all the attribute implications. Rather, it is desirable to have only a relatively small "base" of attribute implications from which all other implications follow. We now formalize these intuitive considerations.

We say that an attribute implication $A \Rightarrow B$ follows from a set $\{A_j \Rightarrow B_j \mid j \in J\}$ of attribute implications $A, B, A_j, B_j \in L^Y$ if, for each fuzzy set $D \in L^Y$, we have that $A \Rightarrow_M B = 1$ (i.e. the truth degree to which $A \Rightarrow B$ is true in $\{M\}$ is 1) whenever for each $j \in J$ we have $A_j \Rightarrow_M B_j = 1$. For a fuzzy context $\langle X, Y, I \rangle$ we say that a set $\mathcal{I} = \{A_j \Rightarrow B_j \mid j \in J\}$ of attribute implications which are true in degree 1 in $\langle X, Y, I \rangle$ forms a *base* for $\langle X, Y, I \rangle$ if each attribute implication $A \Rightarrow B$ which is true in degree 1 in $\langle X, Y, I \rangle$ we have that $A \Rightarrow B$ follows from $\mathcal{I}$. A base $\mathcal{I}$ for $\langle X, Y, I \rangle$ is called irredundant if no $A \Rightarrow B \in \mathcal{I}$ follows from $\mathcal{I} - \{A \Rightarrow B\}$. Therefore, an irredundant base provides complete irredundant information about the attribute implications. Further information about attribute implications (including agorithms) can be found in [Ganter and Wille, 1999] and [Pollandt, 1997].

7.4 Similarity and Logical Precision

Similarity Relations

Similarity phenomenon plays a crucial role in the way humans regard the world. In fact, similarities are induced by the very nature of human perception. Gradual similarity of concepts is one of the fundamental preconditions for powerful human reasoning and communication. Similarity phenomenon is thus one of the most important ones accompanying conceptual structures.

In fuzzy set theory, similarity phenomenon is approached via so called similarity relations (fuzzy equivalence relations), see Chapter 2. For a structure **L** of truth values, a *similarity relation* (fuzzy equivalence) on a set U is a binary fuzzy relation $E : U \times U \to L$ on a universe U satisfying the following properties for all $x, y, z \in U$:

$$E(x, x) = 1 \tag{7.8}$$

$$E(x, y) = E(y, x) \tag{7.9}$$

$$E(x, y) \otimes E(y, z) \leq E(x, z). \tag{7.10}$$

Properties (7.8), (7.9), and (7.10) are called reflexivity, symmetry, and transitivity, respectively. A *similarity class* of $x \in U$ is the fuzzy set $[x]_E \in L^U$ given by $[x]_E(y) = E(x, y)$ for each $y \in U$, i.e. it is a collection of elements similar to

x. A fuzzy set $A \in L^U$ is said to be *compatible* w.r.t. E if for every $x, y \in U$ we have $A(x) \otimes E(x, y) \leq A(y)$. Verbally, this condition says that with each its element x, A contains all the elements similar to x. It is easily seen that in the crisp case, i.e. $L = \{0, 1\}$, similarity relations are equivalence relations. For the study of similarity phenomenon, the crisp case is a degenerate one and non-interesting—two elements x and y may be “fully similar” ($E(x, y) = 1$) or “fully dissimilar” ($E(x, y) = 0$).

Example 7.14 For the three basic structures on $[0, 1]$, i.e. Łukasiewicz, Gödel, and product, transitivity translates to

$$\max(0, E(x, y) + E(y, z) - 1) \leq E(x, z)$$
$$\min(E(x, y), E(y, z)) \leq E(x, z)$$
$$E(x, y) \cdot E(y, z) \leq E(x, z),$$

respectively. Reflexivity and symmetry conditions are the same for each of the three structures.

Note that transitivity expresses a condition which can be formulated in words as “if x and y are similar and if y and z are similar then x and z are similar”. For example, if $E(x, y) = 0.8$ (x and y are similar in degree 0.8) and $E(y, z) = 0.8$ (y and z are similar in degree 0.8) then x and z have to be similar at least in degree $0.8 \otimes 0.8$. Thus, in case of the product structure, transitivity forces $E(x, z) \geq 0.8 \otimes 0.8 = 0.64$.

To model the equivalence (or closeness) of truth values we have at our disposal the so called *biresiduum* (or biimplication) (Pavelka, 1979) operation $\leftrightarrow$ defined by

$$a \leftrightarrow b = (a \to b) \wedge (b \to a).$$

The following lemma will be useful in our considerations:

Lemma 7.15 *Let E be a similarity on U, $\mathcal{S} = \{A_i \in L^U \mid i \in I\}$ be a family of fuzzy sets. (1) E is the largest similarity relation compatible with all $[x]_E$. (2)*

The relation $E_{\mathcal{S}}$ defined by

$$E_{\mathcal{S}}(x,y) = \bigwedge_{i\in I}(A_i(x) \leftrightarrow A_i(y)) \tag{7.11}$$

is the largest similarity relation compatible with all $A_i \in \mathcal{S}$. Moreover, $A_i(x) = 1$ implies $[x]_{E_{\mathcal{S}}} \subseteq A_i$.

Notice that for the crisp case (i.e. $L = \{0,1\}$), $E_{\mathcal{S}}$ is a crisp equivalence relation—two elements of the universe are equivalent if and only if there is no set of the family which separates them.

Remark 7.16 Lemma 7.15 has a very natural interpretation. Each fuzzy set A_i in U represents some property of elements of U — $A_i(x)$ is the degree to which the element x has the property represented by A_i. The degree $A_i(x) \leftrightarrow A_i(y)$ is the truth degree to which "x has the property A_i if and only if y has the property A_i". Therefore, $E_{\mathcal{S}}(x,y)$ is the truth degree of "for each property A from $\mathcal{S}$: x has A if and only if y has A". If $\mathcal{S}$ is the system of all relevant properties, $E_{\mathcal{S}}(x,y)$ is the truth degree to which x and y have the same (relevant) properties.

Example 7.17 For the three basic structures on the real unit interval $[0,1]$, i.e. Łukasiewicz, Gödel, and product, we have

$$E_{\mathcal{S}}(x,y) = \inf_{i\in I}\{1 - |A_i(x) - A_i(y)|\} \quad \text{(Łukasiewicz)}$$

$$E_{\mathcal{S}}(x,y) = \begin{cases} 1 & \text{if } \forall i : A_i(x) = A_i(y) \\ \inf_{i\in I}\{\min(A_i(x), A_i(y))\} & \text{otherwise} \end{cases} \quad \text{(Gödel)}$$

$$E_{\mathcal{S}}(x,y) = \inf_{i\in I}\{\min(A_i(x)/A_i(y), A_i(y)/A_i(x))\} \quad \text{(product)}$$

where we put $0/0 = 1$ and $a/0 = \infty$ for $a \neq 0$.

Example 7.18 We illustrate $E_{\mathcal{S}}$. Let $U = \{x,y,z\}$, $\mathcal{S} = \{A,B\}$ where $A = \{\,{}^{1}/x,\ {}^{0.5}/y,\ {}^{0.1}/z\}$, $B = \{\,{}^{0.9}/x,\ {}^{0.4}/y,\ {}^{0.1}/z\}$. For Łukasiewicz structure on

$[0, 1]$ we get

$$E_{\mathcal{S}}(x, y) = [A(x) \leftrightarrow A(y)] \wedge [B(x) \leftrightarrow B(y)]$$
$$= \quad [1 \leftrightarrow 0.5] \wedge [0.9 \leftrightarrow 0.4] = 0.5 \wedge 0.5 = 0.5,$$

and $E_{\mathcal{S}}(y, z) = 0.6$, $E_{\mathcal{S}}(x, z) = 0.1$. Note that for Gödel structure, we get $E_{\mathcal{S}}(x, y) = 0.4$, $E_{\mathcal{S}}(y, z) = 0.1$, $E_{\mathcal{S}}(x, z) = 0.1$, thus we see that $E_{\mathcal{S}}$ depends on the structure on L.

In the following we consider the problem of similarities on three levels: similarity of objects (and attributes), similarity of attributes, and similarity of concept lattices. The proofs and further results can be found in (Bělohlávek, 2000).

Similarity of Objects and Similarity of Attributes

First, let us propose a way to measure similarity of objects and similarity of attributes of a given fuzzy context. This similarity is induced by the structure of fuzzy concepts determined by the fuzzy context. It turns out that these similarities may be determined directly from the fuzzy context; this is relevant from the computational point of view.

Lemma 7.15 can be directly applied to our problem of measuring similarity of objects and attributes. We are given objects (elements of X) and their observed attributes (elements of Y). A natural question is that of the similarity relation on the objects and attributes. The given fuzzy context gives rise to a complete lattice of all fuzzy concepts hidden in the fuzzy context. As mentioned, the fuzzy concept lattice may be used for the conceptual classification of objects and attributes. It seems therefore reasonable to use the induced conceptual structure $\mathcal{B}(X, Y, I)$ to define similarity relations on X and on Y.

Consider the problem of similarity of objects. Informally, two objects $x_1, x_2 \in X$ are similar if they cannot be separated by any concept, or more precisely, if for each concept c it holds that x_1 belongs to the extent of c if and only if x_2 belongs to the extent of c. This leads to the following definition of a relation $E^X_{\mathcal{B}(X,Y,I)} \in L^{X \times X}$:

$$E^X_{\mathcal{B}(X,Y,I)}(x_1, x_2) = \bigwedge_{\langle A,B\rangle \in \mathcal{B}(X,Y,I)} \Big(A(x_1) \leftrightarrow A(x_2) \Big) \ . \tag{7.12}$$

The relation $E^X_{\mathcal{B}(X,Y,I)}$ will be called *induced* (by $\mathcal{B}(X,Y,I)$) *similarity on* X. By Lemma 7.15 we immediately get the following statement.

Theorem 7.19 *The relation* $E^X_{\mathcal{B}(X,Y,I)}$ *is the largest similarity relation on* X *compatible with the extents of all concepts of* $\mathcal{B}(X,Y,I)$.

From the computational point of view, the foregoing definition leads to the following algorithm for computing the similarity relation $E^X_{\mathcal{B}(X,Y,I)}$. Take a fuzzy context, generate all the fuzzy concepts of $\mathcal{B}(X,Y,I)$ and determine the similarity of each pair $\langle x_1, x_2\rangle \in X \times X$ by (7.12). The fuzzy concept lattice may be, however, quite extensive. This poses the question whether the computational cost can be reduced. An (exact) solution which reduces the computational costs significantly follows. Define a relation $E^X_{\langle X,Y,I\rangle} \in L^{X \times X}$ by

$$E^X_{\langle X,Y,I\rangle}(x_1, x_2) = \bigwedge_{y \in Y} \left(I(x_1, y) \leftrightarrow I(x_2, y) \right) \ . \tag{7.13}$$

$E^X_{\langle X,Y,I\rangle}(x_1, x_2)$ may be obtained from the L-context $\langle X, Y, I\rangle$ computing $|Y|$ times the operation $\leftrightarrow$. Using Lemma 7.15 (put $X = X$, $I = Y$, $A_i(x) = I(x, y)$) we get the following theorem.

Theorem 7.20 *The relation* $E^X_{\langle X,Y,I\rangle}$ *is the largest similarity relation on* X *compatible with all* $I(\_, y) \in L^X$, $y \in Y$.

The following theorem solves the problem of finding an efficient procedure for computing the similarity relation $E^X_{\mathcal{B}(X,Y,I)}$.

Theorem 7.21 *Let* $\langle X, Y, I\rangle$ *be a fuzzy context. Then for the similarity relations defined by (7.13) and (7.12) we have*

$$E^X_{\mathcal{B}(X,Y,I)} = E^X_{\langle X,Y,I\rangle} \ . \tag{7.14}$$

Hence, the computation of $E^X_{\mathcal{B}(X,Y,I)}$ may be reduced to the computation of $E^X_{\langle X,Y,I\rangle}$ which is much more simple.

In a completely analogous way we may get the results for the similarity relations on Y.

Similarity of Concepts

The next level on which the similarity phenomenon will be considered is the level of concepts. Observe first the following fact which follows from Lemma 7.15.

Lemma 7.22 *For any universe U, the relation E on L^U given for any $A_1, A_2 \in L^U$ by*

$$E(A_1, A_2) = \bigwedge_{x \in U} (A_1(x) \leftrightarrow A_2(x))$$

is the largest similarity relation on L^U such that $A_1(x) \otimes E(A_1, A_2) \le A_2(x)$ holds for each $x \in U$, $A_1, A_2 \in L^U$.

$E(A_1, A_2)$ is thus the truth degree to which it is true that x belongs to A_1 if and only if x belongs to A_2. In the following, it will be clear what universe U the relation E concerns.

Example 7.23 Let again $U = \{x, y, z\}$, $\mathcal{S} = \{A, B\}$ where $A = \{\,{}^{1}/x,\ {}^{0.5}/y,\ {}^{0.1}/z\}$, $B = \{\,{}^{0.9}/x,\ {}^{0.4}/y,\ {}^{0.1}/z\}$. For Łukasiewicz structure on $[0, 1]$ we get

$$E(A, B) = [A(x) \leftrightarrow B(x)] \wedge [A(y) \leftrightarrow B(y)] \wedge [A(z) \leftrightarrow B(z)]$$
$$= [1 \leftrightarrow 0.9] \wedge [0.5 \leftrightarrow 0.4] \wedge [0.1 \leftrightarrow 0.1] = 0.9 \wedge 0.9 \wedge 1 = 0.9.$$

For Gödel structure we analogously have $E(A, B) = 0.4$, while for the product structure we have $E(A, B) = 0.8$.

Consider first the relations E^{Ext} and E^{Int} on $\mathcal{B}(X, Y, I)$, call them *induced similarity* by extents and *induced similarity* by intents, respectively:

$$E^{Ext}(\langle A_1, B_1\rangle, \langle A_2, B_2\rangle) = E(A_1, A_2) = \bigwedge_{x \in X} (A_1(x) \leftrightarrow A_2(x)),$$

$$E^{Int}(\langle A_1, B_1\rangle, \langle A_2, B_2\rangle) = E(B_1, B_2) = \bigwedge_{y \in Y} (B_1(y) \leftrightarrow B_2(y)).$$

Lemma 7.22 gives immediately the following statement.

Theorem 7.24 *E^{Ext} and E^{Int} are the largest similarity relations on $\mathcal{B}(X, Y, I)$ such that $A_1(x) \otimes E^{Ext}(\langle A_1, B_1\rangle, \langle A_2, B_2\rangle) \leq A_2(x)$ and $B_1(y) \otimes E^{Int}(\langle A_1, B_1\rangle, \langle A_2, B_2\rangle) \leq B_2(y)$ hold for every $x \in X$, $y \in Y$, $\langle A_1, B_1\rangle$, $\langle A_2, B_2\rangle \in \mathcal{B}(X, Y, I)$.*

The following theorem answers the question of how the relations E^{Ext} and E^{Int} are related.

Theorem 7.25 *For any fuzzy context $\langle X, Y, I\rangle$ we have $E^{Ext} = E^{Int}$.*

We can therefore write E instead of E^{Ext} and E^{Int} and call it the *induced similarity* on concepts.

Compatible Similarities and Factorization

The primary importance of similarity relations in human reasoning is the *reduction of the complexity of the world at a reasonable price*. The complexity is reduced via considering the "collections of similar elements of concern" rather than the particular elements themselves. This is known in general system theory as the *abstraction process by factorization*: moving from a given level of abstraction (distinguishability) one level up where the elements are collections of elements of the lower level. Instead of the original system one therefore considers the "system modulo similarity". The price paid is the loss of precision.

Our concern in the following is the *reduction of the complexity of the concept lattice by factorization modulo similarity*. The concept lattice of a given context represents the overall conceptual structure which can be considerably intricate. To get an insight one has to look for methods for reducing the complexity of the structure. In the two-valued (crisp) case, considerable attention has been paid to this problem [Ganter and Wille, 1999]. In the fuzzy case, one would expect methods for gradual reduction of the complexity. The idea is to factorize the concept lattice by appropriate α-cut ${}^{\alpha}E$ of the similarity E (note that ${}^{\alpha}E = \{\langle c_1, c_2\rangle \mid \alpha \leq E(c_1, c_2)\}$), controlling thus the complexity by $\alpha \in L$. Clearly, the lower $\alpha \in L$, the coarser the factorization. The process of factorization of a system consists in two steps. First, specification of the elements, and, secondly, specification of the structure of the factor system. Since both of the steps are non-standard in our case we will describe them in more detail. In general, algebraic systems can be factorized be *congruences*, i.e. equivalences compatible with the structure of the system. We deal with conceptual structures which are complete lattices. The a-cut ${}^{a}E$ is clearly a tolerance relation (i.e. reflexive and symmetric), not transitive in general. In general, factorization of algebras by compatible tolerances is not possible. Surprisingly, [Czédli, 1982] showed a way to factorize lattices by compatible tolerance relations. The construction has been then used for the factorization of ordinary concept lattices [Wille, 1985]. In the following, we describe the construction of the factor lattice of a fuzzy concept lattice by a compatible tolerance relation. Let $\langle X, Y, I\rangle$ be a fuzzy context. A tolerance relation T on $\mathcal{B}(X, Y, I)$ is said to be *compatible* if it is preserved under arbitrary suprema and infima, i.e. if $\langle c_j, c'_j\rangle \in T$, $j \in J$, implies both $\langle \bigvee_{j\in J} c_j, \bigvee_{j\in J} c'_j\rangle \in T$ and $\langle \bigwedge_{j\in J} c_j, \bigwedge_{j\in J} c'_j\rangle \in T$ for any $c_j, c'_j \in \mathcal{B}(X, Y, I)$, $j \in J$. For a compatible tolerance relation T on $\mathcal{B}(X, Y, I)$ denote $c_T = \bigwedge_{\langle c,c'\rangle\in T} c'$ and $c^T = \bigvee_{\langle c,c'\rangle\in T} c'$. Call $[c]_T = [c_T, (c_T)^T] = \{c' \in \mathcal{B}(X, Y, I) \mid c_T \leq c' \leq (c_T)^T\}$ a *block* of T and denote $\mathcal{B}(X, Y, I)/T = \{[c]_T \mid c \in \mathcal{B}(X, Y, I)\}$ the set of all blocks. Introduce a relation $\leq_T$ on $\mathcal{B}(X, Y, I)/T$ by $[c]_T \leq_T [c']_T$ if and only if $\bigwedge[c]_T \leq \bigwedge[c']_T$ (if and only if $\bigvee[c]_T \leq \bigvee[c']_T$). The justification of the construction is given by the following statement which follows immediately from [Wille, 1985].

Theorem 7.26 *(1)* $\mathcal{B}(X, Y, I)/T$ *is the set of all maximal tolerance blocks, i.e.* $\mathcal{B}(X, Y, I)/T = \{B \subseteq \mathcal{B}(X, Y, I) \mid (B \times B \subseteq T)\&((\forall B' \supset B)B' \times B' \not\subseteq T)\}$. *(2)* $\langle \mathcal{B}(X, Y, I)/T, \leq_T\rangle$ *is a complete lattice* (factor lattice) *where suprema and* *infima are described by*

$$\bigvee_{j\in J}[c_j]_T = [\bigvee_{j\in J} c_j]_T \quad \textit{and} \quad \bigwedge_{j\in J}[c_j]_T = [(\bigwedge_{j\in J} c_j)^T]_T \tag{7.15}$$

for every $c_j \in \mathcal{B}(X,Y,I)$*,* $j \in J$*.*

Substituting (7.4) and (7.5) into (7.15) we get a more concrete description of the lattice operations.

Coming back to the induced similarity E on $\mathcal{B}(X,Y,I)$, the ultimate question is that of the compatibility of the a-cuts of E. Call a similarity relation F on $\mathcal{B}(X,Y,I)$ *compatible* if ${}^{\alpha}E$ is a compatible tolerance relation on $\mathcal{B}(X,Y,I)$ for each $\alpha \in L$. Notice that for the two-valued (crisp) case the situation is completely uninteresting. Namely, as one easily checks, the only cases are ${}^{0}E = \mathcal{B}(X,Y,I) \times \mathcal{B}(X,Y,I)$ and ${}^{1}E = \mathrm{id}_{\mathcal{B}(X,Y,I)} = \{\langle c,c\rangle \mid c \in \mathcal{B}(X,Y,I)\}$. In the first case, $\mathcal{B}(X,Y,I)/{}^{0}E = \{\mathcal{B}(X,Y,I)\}$, i.e. the factor lattice collapsed into a one element lattice, while in the second case, $\mathcal{B}(X,Y,I)/{}^{1}E = \{\{\langle A,B\rangle\} \mid \langle A,B\rangle \in \mathcal{B}(X,Y,I)\}$, i.e. $\mathcal{B}(X,Y,I)$ and $\mathcal{B}(X,Y,I)/{}^{1}E$ are isomorphic.

Note that we need not confine ourselves to the induced similarity E. On the other hand, taking into account only similarity relations F satisfying $A(x) \otimes F(\langle A,B\rangle, \langle A',B'\rangle) \leq A'(x)$ (which is quite natural—it reads "object belonging to the extent of some concept belongs also to the extent of any similar concept") for each $x \in X$, Theorem 7.24 tells us that E provides the most extensive reduction: for any other F and each $a \in L$, ${}^{a}E$ is coarser than ${}^{a}F$.

Theorem 7.27 *The induced similarity* E *on* $\mathcal{B}(X,Y,I)$ *is compatible. If* $\alpha \in L$ *is* $\otimes$*-idempotent (i.e.* $\alpha \otimes \alpha = \alpha$*) then* ${}^{\alpha}E$ *is, moreover, transitive, i.e. a congruence relation on* $\mathcal{B}(X,Y,I)$*.*

Remark 7.28 Theorem 7.27 and the above described construction yield a method for factorizing any fuzzy concept lattice $\mathcal{B}(X,Y,I)$ by any α-cut ${}^{\alpha}E$ of the induced similarity E. It is worth to notice that the similarity E is defined "internally", i.e. it is not supplied from the outside.

Remark 7.29 If **L** the Gödel algebra on $[0,1]$, i.e. $\otimes$ is min, then each α-cut of E is indeed a congruence relation.

Similarity of Concept Lattices

Finally, we consider similarity of concept lattices. A natural way to define the similarity degree of two concept lattices over the sets X and Y is based on the following intuition. Concept lattices $\mathcal{B}(X,Y,I_1)$ and $\mathcal{B}(X,Y,I_2)$ are similar if and only if for each concept $c_1 \in \mathcal{B}(X,Y,I_1)$ there is a concept $c_2 \in \mathcal{B}(X,Y,I_2)$ such that c_1 and c_2 are similar and, conversely, for each concept $c_2 \in \mathcal{B}(X,Y,I_2)$ there is a concept $c_1 \in \mathcal{B}(X,Y,I_1)$ such that c_1 and c_2 are similar. In the following we write $\mathcal{B}_1$ and $\mathcal{B}_2$ instead of $\mathcal{B}(X,Y,I_1)$ and $\mathcal{B}(X,Y,I_2)$, respectively. According to how the similarity of concepts is measured we distinguish two rules for the definition of the similarity degree of two concept lattices:

$$\begin{aligned} E^*(\mathcal{B}(X,Y,I_1),\mathcal{B}(X,Y,I_2)) = \\ = \bigwedge_{\langle A_1,B_1\rangle\in\mathcal{B}_1} \bigvee_{\langle A_2,B_2\rangle\in\mathcal{B}_2} E^*(\langle A_1,B_1\rangle,\langle A_2,B_2\rangle) \wedge \\ \bigwedge_{\langle A_2,B_2\rangle\in\mathcal{B}_2} \bigvee_{\langle A_1,B_1\rangle\in\mathcal{B}_1} E^*(\langle A_1,B_1\rangle,\langle A_2,B_2\rangle), \end{aligned} \tag{7.16}$$

$* \in \{Ext, Int\}$, where we put

$$\begin{aligned} E^{Ext}(\langle A_1,B_1\rangle,\langle A_2,B_2\rangle) &= E(A_1,A_2) \\ E^{Int}(\langle A_1,B_1\rangle,\langle A_2,B_2\rangle) &= E(B_1,B_2). \end{aligned}$$

We shall see that all of the above relations are in fact similarity relations. To investigate their relationships to the similarity relation E defined on the set of all contexts by

$$\begin{aligned} E(\langle X,Y,I_1\rangle,\langle X,Y,I_2\rangle) = \\ = E(I_1,I_2) = \bigwedge_{\langle x,y\rangle\in X\times Y} I_1(x,y) \leftrightarrow I_2(x,y). \end{aligned} \tag{7.17}$$

It follows directly from Lemma 7.22 immediately gives that E is a similarity relation. The main result showing the relationships between the above introduced relations is contained in the following theorem.

Theorem 7.30 *For every fuzzy contexts $\langle X, Y, I_1 \rangle$, $\langle X, Y, I_2 \rangle$ we have*

$$E(\langle X, Y, I_1 \rangle, \langle X, Y, I_2 \rangle) \leq E^{Ext}(\mathcal{B}_1, \mathcal{B}_2)$$

and

$$E(\langle X, Y, I_1 \rangle, \langle X, Y, I_2 \rangle) \leq E^{Int}(\mathcal{B}_1, \mathcal{B}_2).$$

Moreover, E^{Ext} and E^{Int} are similarity relations on $\{\mathcal{B}(X, Y, I) \mid I \in L^{X \times Y}\}$.

Logical Precision

Consider a structure $\mathbf{L}$ of truth values. The set L is the set of all possible truth values which we have at our disposal for logical modeling of our knowledge. It can be considered as representing "logical discernibility". Consider for example the two-element Boolean algebra. Then the level of discernibility is low—we can discern only fully true statements from fully false statements. An n-element chain of truth values offers more—we can discern n logical "levels". Very loosely, using more truth values means more logical precision (in the above sense). From the point of view of logical modeling, it is natural to be able to change the set of truth values (in order to increase or decrease the logical discernibility) so that the structural properties of the model remain preserved. An important role is played by the structure of the set of truth values which has to be reflected. Consider two structures $\mathbf{L}_1$ and $\mathbf{L}_2$ of truth values such that there is a surjective mapping $h : L_1 \to L_2$, i.e. $h(L_1) = L_2$. If h preserves the structure of the sets of truth values then the change from L_2 to L_1 can be considered as an increase of logical precision and, conversely, change from L_1 to L_2 can be considered as an decrease of logical precision. The requirement of preserving the structure of truth values may be, from the algebraic point of view, seen as fulfilled if h is a homomorphism (Grätzer, 1968). In our case, h is a *homomorphism* if and only if the following conditions are satisfied:

$$h(a \vee b) = h(a) \vee h(b)$$
$$h(a \wedge b) = h(a) \wedge h(b)$$
$$h(a \otimes b) = h(a) \otimes h(b)$$
$$h(a \rightarrow b) = h(a) \rightarrow h(b)$$
$$h(0) = 0$$
$$h(1) = 1.$$

In the following we will suppose that all the homomorphisms under consideration will be $\bigwedge$-preserving, i.e. for each $K \subseteq L_1$ we have $h(\bigwedge_{k \in K} k) = \bigwedge_{k \in K} h(k)$. Given two structures $\mathbf{L}_1$ and $\mathbf{L}_2$ of truth values and a homomorpism $h : \mathbf{L}_1 \rightarrow \mathbf{L}_2$, we define for each $\mathbf{L}_1$-fuzzy set A in X ($A \in L_1^X$) the corresponding $\mathbf{L}_2$-fuzzy set $h(A) \in L_2^X$ by $(h(A))(x) = h(A(x))$ for all $x \in X$. The following two statements show how the systematic change of the set of truth values (i.e. increase or decrease of logical precision) influences the structure of the respective concepts (Bělohlávek, 2002).

Lemma 7.31 *Let $\mathbf{L}_1$, $\mathbf{L}_2$ be complete residuated lattices and $h : \mathbf{L}_1 \rightarrow \mathbf{L}_2$ be an onto homomorphism. Let $\langle X, Y, I \rangle$ be an $\mathbf{L}_1$-context. Then for $C \in L_2^X$, $D \in L_2^Y$, the following holds: $\langle C, D \rangle \in \mathcal{B}(X, Y, h(I))$ iff there are $A \in L_1^X$, $B \in L_1^Y$ such that $\langle A, B \rangle \in \mathcal{B}(X, Y, I)$, $h(A) = C$, and $h(B) = D$.*

A lattice homomorphism $h : \mathbf{V}_1 \rightarrow \mathbf{V}_2$ between two complete lattices $\mathbf{V}_1$ and $\mathbf{V}_2$ is called *complete* if for each $K \subseteq V_1$ we have $h(\bigwedge_{k \in K}) = \bigwedge_{k \in K} h(k)$ and $h(\bigvee_{k \in K}) = \bigvee_{k \in K} h(k)$.

Theorem 7.32 *Under the conditions of the preceding lemma, the mapping h^* as defined by*

$$h^*(A, B) = \langle h(A), h(B) \rangle$$

is a complete homomorphism of $\mathcal{B}(X, Y, I)$ onto $\mathcal{B}(X, Y, h(I))$.

Remark 7.33 The foregoing theorem is relevant from the application point of view: Suppose we have a concept with truth values from L_1. A further analysis on the level of L_1 may be (from various reasons, e.g. computational ones) "too precise". We can then skip to a level of $L_2 = h(L_1)$ which is appropriate. Due to the theorem, the structure of the concepts changes systematically, i.e. the structure of concepts in L_1 is in a systematic way more precise than that one in L_2.

There is an important consequence of Theorem 7.32: The fuzzy concept lattice $\mathcal{B}(X, Y, h(I))$ can be thought of as if obtained from $\mathcal{B}(X, Y, I)$ by factorization, i.e. by the process of abstraction. Namely, the mapping h^* of $\mathcal{B}(X, Y, I)$ to $\mathcal{B}(X, Y, h(I))$ induces an equivalence relation θ_{h^*} on $\mathcal{B}(X, Y, I)$ by

$$\langle\langle A_1, B_1\rangle, \langle A_2, B_2\rangle\rangle \in \theta_{h^*} \quad \text{iff} \quad h^*(A_1, B_1) = h^*(A_2, B_2).$$

That is we can consider a so-called factor set $\mathcal{B}(X, Y, I)/\theta_{h^*}$ of $\mathcal{B}(X, Y, I)$ modulo θ_{h^*}. The elements of $\mathcal{B}(X, Y, I)/\theta_{h^*}$ are classes $[\langle A, B\rangle]_{\theta_{h^*}}$ of pairwise θ_{h^*}-equivalent fuzzy concepts, i.e.

$$[\langle A, B\rangle]_{\theta_{h^*}} = \{\langle A', B'\rangle \in \mathcal{B}(X, Y, I) \mid \langle\langle A, B\rangle, \langle A', B'\rangle\rangle \in \theta_{h^*}\}.$$

Now, θ_{h^*} is moreover a complete congruence on $\mathcal{B}(X, Y, I)$. This means that one can define operations on $\mathcal{B}(X, Y, I)/\theta_{h^*}$ in such a way that it becomes a complete lattice: we put

$$\begin{aligned}
&[\langle A_1, B_1\rangle]_{\theta_{h^*}} \wedge [\langle A_2, B_2\rangle]_{\theta_{h^*}} = [\langle A_1, B_1\rangle \wedge \langle A_2, B_2\rangle]_{\theta_{h^*}}\\
&[\langle A_1, B_1\rangle]_{\theta_{h^*}} \vee [\langle A_2, B_2\rangle]_{\theta_{h^*}} = [\langle A_1, B_1\rangle \vee \langle A_2, B_2\rangle]_{\theta_{h^*}}\\
&\bigwedge_{i\in I}[\langle A_i, B_i\rangle]_{\theta_{h^*}} = [\bigwedge_{i\in I} \langle A_i, B_i\rangle]_{\theta_{h^*}}\\
&\bigvee_{i\in I}[\langle A_i, B_i\rangle]_{\theta_{h^*}} = [\bigvee_{i\in I} \langle A_i, B_i\rangle]_{\theta_{h^*}}.
\end{aligned}$$

The following theorem follows from elementary facts of general algebra.

Theorem 7.34 $\mathcal{B}(X, Y, I)/\theta_{h^*}$ *equipped with the above defined operations is isomorphic to* $\mathcal{B}(X, Y, h(I))$.

This means that $\mathcal{B}(X, Y, I)/\theta_{h^*}$ and $\mathcal{B}(X, Y, h(I))$ do differ only in relabeling their elements. That is, $\mathcal{B}(X, Y, h(I))$ may be seen as if obtained from $\mathcal{B}(X, Y, I)$ by abstraction.

7.5 Formal Concept Analysis Demonstrated: Examples

Example 7.35 Our main purpose is to illustrate methods described above. The example is taken from paleontology and it is a simplified version of a larger example examined in a forthcoming paper by Bělohlávek and Košt'ák. The input data are fossils outlines depicted in Figure 7.5.

Each of the fossils consists of a body and a spin. Intuitively, the fossils belong to some general category C (covering them all). Suppose we have no previous knowledge about what natural categories (subcategories of C) exist. A first approximation of natural subcategories of C may be automatically extracted from the input data using formal concept analysis.

The first step is to write down an appropriate context from the input data. That is, we have to identify objects and (fuzzy) attributes. We naturally take the nine fossils for the objects; we assign numbers 1–9 to them, according to Figure 7.5. Now, we have to identify suitable (fuzzy) attributes. Needless to say, identification of attributes of the fossils is an arbitrary process. What makes the fossils look different here are basically two features: first, the size of the spine and, second, the shape of the body. Concerning the first feature, we identify two attributes, "spine small" and "spine big". These attributes are naturally fuzzy ones. Concerning the shape of the body, the shape goes from circle-shaped to very much oval-shaped. The key feature is thus the ratio length:width; we identify two attributes, "oval-shaped" (length:width big) and "circle-shaped" (length:width small). The attributes and their meaning are summarized in Table 7.5.

We now have to take an appropriate set of truth values and equip it with an appropriate structure. We take $L = \{0, \frac{1}{2}, 1\}$ and will consider two structures defined on L, the Łukasiewicz one and the Gödel one. The context is given by

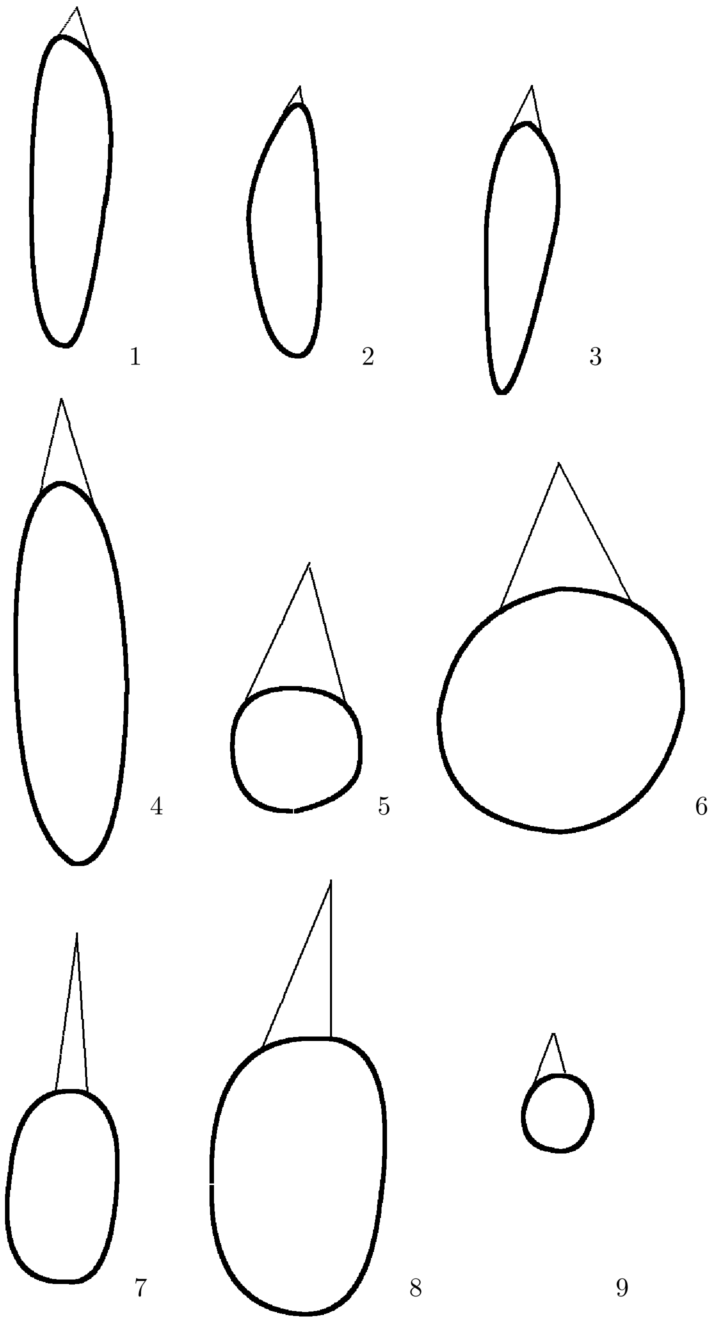

Figure 7.2: Fossils from Example 7.35.

abbreviation	meaning
ss	has a small spine
sb	has a big spine
cs	is circle-shaped
os	is oval-shaped

Table 7.4: Attributes of fossils and their meaning.

	spine small ss	spine big sb	circle-shaped cs	oval-shaped os
fossil 1	1	0	0	1
fossil 2	1	0	0	1
fossil 3	1	0	0	1
fossil 4	1	0	$\frac{1}{2}$	1
fossil 5	0	1	1	$\frac{1}{2}$
fossil 6	0	1	1	$\frac{1}{2}$
fossil 7	$\frac{1}{2}$	$\frac{1}{2}$	1	0
fossil 8	$\frac{1}{2}$	$\frac{1}{2}$	1	0
fossil 9	1	0	1	0

Table 7.5: Fuzzy context given by fossils and their properties.

the Table 7.5. Therefore, the set X of objects contains nine elements (denoted by 1,…,9); the set Y of attribute contains four elements (denoted ss, sb, cs, os).

The corresponding fuzzy concept lattices are depicted in Figure 7.3 (Łukasiewicz structure on L) and Figure 7.4 (Gödel structure on L).

To gain more insight, the elements (i.e. concepts) of the lattice are identified in Table 7.6 (Łukasiewicz structure) and Table 7.7 (Gödel structure).

Table 7.8 shows the similarity relation $E^{X}_{\mathcal{B}(X,Y,I)}$ on the fossils.

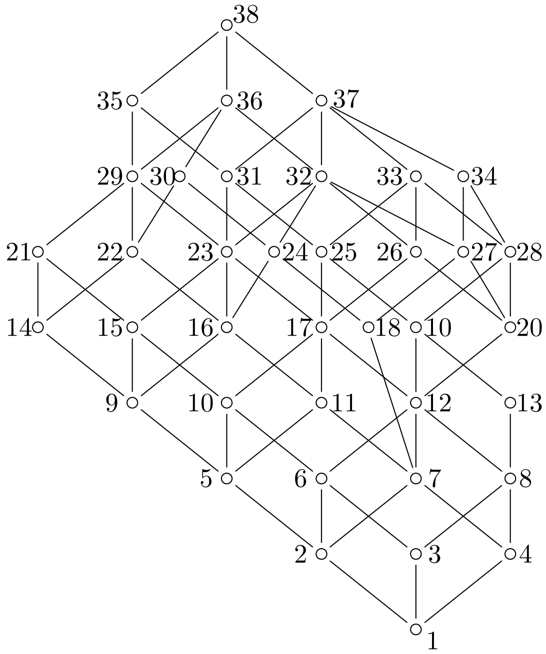

Figure 7.3: Fuzzy concept lattice of the context in Table 7.5 (Łukasiewicz structure).

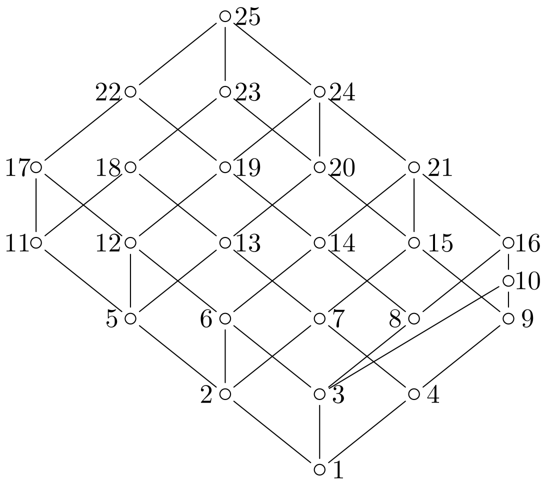

Figure 7.4: Fuzzy concept lattice of the context in Table 7.5 (Gödel structure).

no.	extent									intent			
	1	2	3	4	5	6	7	8	9	ss	sb	cs	os
1.	0	0	0	0	0	0	0	0	0	1	1	1	1
2.	0	0	0	$\frac{1}{2}$	0	0	0	0	0	1	$\frac{1}{2}$	1	1
3.	0	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	$\frac{1}{2}$	1	1	1
4.	0	0	0	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0	1	1	1	$\frac{1}{2}$
5.	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	0	0	1	$\frac{1}{2}$	$\frac{1}{2}$	1
6.	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	1	1
7.	0	0	0	$\frac{1}{2}$	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	1	$\frac{1}{2}$
8.	0	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	1	1	$\frac{1}{2}$
9.	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	0	0	0	0	0	1	0	$\frac{1}{2}$	1
10.	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1
11.	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
12.	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$
13.	0	0	0	0	1	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	1	1	$\frac{1}{2}$
14.	1	1	1	1	0	0	0	0	0	1	0	0	1
15.	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	$\frac{1}{2}$	0	$\frac{1}{2}$	1
16.	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	0	$\frac{1}{2}$	$\frac{1}{2}$
17.	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
18.	0	0	0	$\frac{1}{2}$	0	0	$\frac{1}{2}$	$\frac{1}{2}$	1	1	0	1	0
19.	0	0	0	$\frac{1}{2}$	1	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{1}{2}$
20.	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	0
21.	1	1	1	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	$\frac{1}{2}$	0	0	1
22.	1	1	1	1	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	0	0	$\frac{1}{2}$
23.	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$
24.	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	0	0	$\frac{1}{2}$	$\frac{1}{2}$	1	1	0	$\frac{1}{2}$	0
25.	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
26.	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0
27.	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	1	1	$\frac{1}{2}$	0	1	0
28.	0	0	0	$\frac{1}{2}$	1	1	1	1	$\frac{1}{2}$	0	$\frac{1}{2}$	1	0
29.	1	1	1	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{1}{2}$
30.	1	1	1	1	0	0	$\frac{1}{2}$	$\frac{1}{2}$	1	1	0	0	0
31.	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	1	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{1}{2}$	$\frac{1}{2}$
32.	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	1	1	1	$\frac{1}{2}$	0	$\frac{1}{2}$	0
33.	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	1	1	1	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	0
34.	0	0	0	$\frac{1}{2}$	1	1	1	1	1	0	0	1	0
35.	1	1	1	1	1	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	$\frac{1}{2}$
36.	1	1	1	1	$\frac{1}{2}$	$\frac{1}{2}$	1	1	1	$\frac{1}{2}$	0	0	0
37.	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	1	1	1	1	1	0	0	$\frac{1}{2}$	0
38.	1	1	1	1	1	1	1	1	1	0	0	0	0

Table 7.6: Fuzzy concepts of the context of Table 7.5 (Łukasiewicz structure).

no.	extent									intent			
	1	2	3	4	5	6	7	8	9	ss	sb	cs	os
1.	0	0	0	0	0	0	0	0	0	1	1	1	1
2.	0	0	0	$\frac{1}{2}$	0	0	0	0	0	1	0	1	1
3.	0	0	0	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0	1	1	1	0
4.	0	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	0	1	1	1
5.	0	0	0	1	0	0	0	0	0	1	0	$\frac{1}{2}$	1
6.	0	0	0	$\frac{1}{2}$	0	0	$\frac{1}{2}$	$\frac{1}{2}$	1	1	0	1	0
7.	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	0	0	1	1
8.	0	0	0	0	0	0	1	1	0	$\frac{1}{2}$	$\frac{1}{2}$	1	0
9.	0	0	0	0	1	1	0	0	0	0	1	1	$\frac{1}{2}$
10.	0	0	0	0	1	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	1	1	0
11.	1	1	1	1	0	0	0	0	0	1	0	0	1
12.	0	0	0	1	0	0	$\frac{1}{2}$	$\frac{1}{2}$	1	1	0	$\frac{1}{2}$	0
13.	0	0	0	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	0	0	$\frac{1}{2}$	1
14.	0	0	0	$\frac{1}{2}$	0	0	1	1	1	$\frac{1}{2}$	0	1	$\frac{1}{2}$
15.	0	0	0	$\frac{1}{2}$	1	1	0	0	0	0	0	1	$\frac{1}{2}$
16.	0	0	0	0	1	1	1	1	0	0	$\frac{1}{2}$	1	0
17.	1	1	1	1	0	0	$\frac{1}{2}$	$\frac{1}{2}$	1	1	0	0	0
18.	1	1	1	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	0	0	0	1
19.	0	0	0	1	0	0	1	1	1	$\frac{1}{2}$	0	$\frac{1}{2}$	0
20.	0	0	0	1	1	1	0	0	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$
21.	0	0	0	$\frac{1}{2}$	1	1	1	1	1	0	0	1	0
22.	1	1	1	1	0	0	1	1	1	$\frac{1}{2}$	0	0	0
23.	1	1	1	1	1	1	0	0	0	0	0	0	$\frac{1}{2}$
24.	0	0	0	1	1	1	1	1	1	0	0	$\frac{1}{2}$	0
25.	1	1	1	1	1	1	1	1	1	0	0	0	0

Table 7.7: Fuzzy concepts of the context of Table 7.5 (Gödel structure).

	fos. 1	fos. 2	fos. 3	fos. 4	fos. 5	fos. 6	fos. 7	fos. 8	fos. 9
fossil 1	1	1	1	$\frac{1}{2}$	0	0	0	0	0
fossil 2		1	1	$\frac{1}{2}$	0	0	0	0	0
fossil 3			1	$\frac{1}{2}$	0	0	0	0	0
fossil 4				1	0	0	0	0	0
fossil 5					1	1	$\frac{1}{2}$	$\frac{1}{2}$	0
fossil 6						1	1	$\frac{1}{2}$	0
fossil 7							1	1	$\frac{1}{2}$
fossil 8								1	$\frac{1}{2}$
fossil 9									1

Table 7.8: Similarity $E^X_{\mathcal{B}(X,Y,I)}$ on fossils from Figure 7.5 and Table 7.5.

We now illustrate reduction of a fuzzy concept lattice by decrease of logical precision (cf. Section 7.4). Consider the Gödel structure $\mathbf{L}$ on L and the mapping $h : L \to \{0, 1\}$ defined by

$$h : \begin{array}{ccc} 0 & \mapsto & 0 \\ \frac{1}{2} & \mapsto & 1 \\ 1 & \mapsto & 1. \end{array}$$

It is easy to verify that h is a ($\bigwedge$-preserving) homomorphism of $\mathbf{L}$ onto $\mathbf{2}$. The context $\langle X, Y, h(I)\rangle$ is depicted in Table 7.9.

According to Theorem 7.32 and Theorem 7.34, the mapping h^* sending $\langle A, B\rangle \in \mathcal{B}(X, Y, I)$ to $\langle h(A), h(B)\rangle$ is a complete homomorphism of $\mathcal{B}(X, Y, I)$ onto $\mathcal{B}(X, Y, h(I))$. This mapping induces a congruence θ_{h^*} on $\mathcal{B}(X, Y, I)$ so that two fuzzy concepts $\langle A_1, B_1\rangle$ and $\langle A_2, B_2\rangle$ from $\mathcal{B}(X, Y, I)$ belong to the same class of θ_{h^*} (are θ_{h^*}-congruent) iff $h(A_1) = h(A_2)$ and $h(B_1) = h(B_2)$. The classes of θ_{h^*} are depicted in Figure 7.5. The elements of $\mathcal{B}(X, Y, h(I))$ corresponding to the classes of θ_{h^*} are listed in Table 7.10.

The lattice $\mathcal{B}(X, Y, h(I))$ (which is isomorphic to $\mathcal{B}(X, Y, I)/\theta_{h^*}$) is depicted in Figure 7.7.

	spine small ss	spine big sb	circle-shaped cs	oval-shaped os
fossil 1	1	0	0	1
fossil 2	1	0	0	1
fossil 3	1	0	0	1
fossil 4	1	0	1	1
fossil 5	0	1	1	1
fossil 6	0	1	1	1
fossil 7	1	1	1	0
fossil 8	1	1	1	0
fossil 9	1	0	1	0

Table 7.9: Context $\langle X, Y, h(I)\rangle$ corresponding to $\langle X, Y, I\rangle$ from Table 7.5.

no.	extent									intent			
	1	2	3	4	5	6	7	8	9	ss	sb	cs	os
1.	0	0	0	0	0	0	0	0	0	1	1	1	1
2.	0	0	0	1	0	0	0	0	0	1	0	1	1
3.	0	0	0	0	0	0	1	1	0	1	1	1	0
4.	0	0	0	0	1	1	0	0	0	0	1	1	1
5.	1	1	1	1	0	0	0	0	0	1	0	0	1
6.	0	0	0	1	0	0	1	1	1	1	0	1	0
7.	0	0	0	1	1	1	0	0	0	0	0	1	1
8.	0	0	0	0	1	1	1	1	0	0	1	1	0
9.	1	1	1	1	0	0	1	1	1	1	0	0	0
10.	1	1	1	1	1	1	0	0	0	0	0	0	1
11.	0	0	0	1	1	1	1	1	1	0	0	1	0
12.	1	1	1	1	1	1	1	1	1	0	0	0	0

Table 7.10: Concepts of $\mathcal{B}(X, Y, h(I))$.

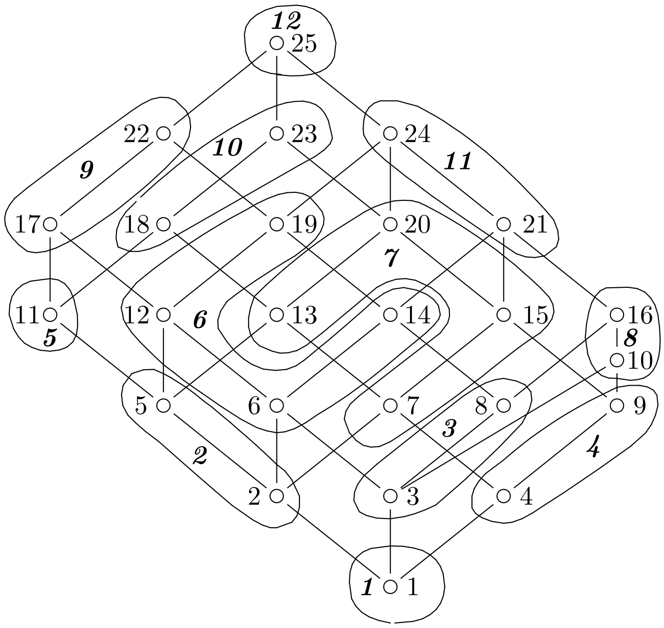

Figure 7.5: Classes of the congruence relation induced by h.

Consider now the Łukasiewicz structure on $\{0, \frac{1}{2}, 1\}$. We illustrate the factorization by similarity. Consider the a-cut of the induced similarity E on $\mathcal{B}(X, Y, I)$ for $\alpha = \frac{1}{2}$, i.e. consider ${}^{\frac{1}{2}}E$. The tolerance blocks in $\mathcal{B}(X, Y, I)$ (which are, in fact, complete sublattices) are depicted in Figure 7.6. Note that each block is a maximal subset of **L**-concepts which are similar in the degree at least $\frac{1}{2}$. The corresponding factor lattice $\mathcal{B}(X, Y, I)/{}^{\frac{1}{2}}E$ is depicted in Figure 7.7.

A few remarks to the examples. There are apparently natural concepts: Consider e.g. the Łukasiewicz structure. Concept no. 14 is naturally described as a fossil with a small spine and an oval shape. Concept no. 26 is "a fossil with a circle shape which has a bit small spine". Concept no. 1 is an example of an (empirically) empty concept (its extent is an empty set). Concept no. 17 (and also al the concepts between 1 and 17, i.e. 2, 3, 4, 5, 6, 7, 8, 10, 11, 12), however, do not contain any object in degree 1. One may thus wish to consider

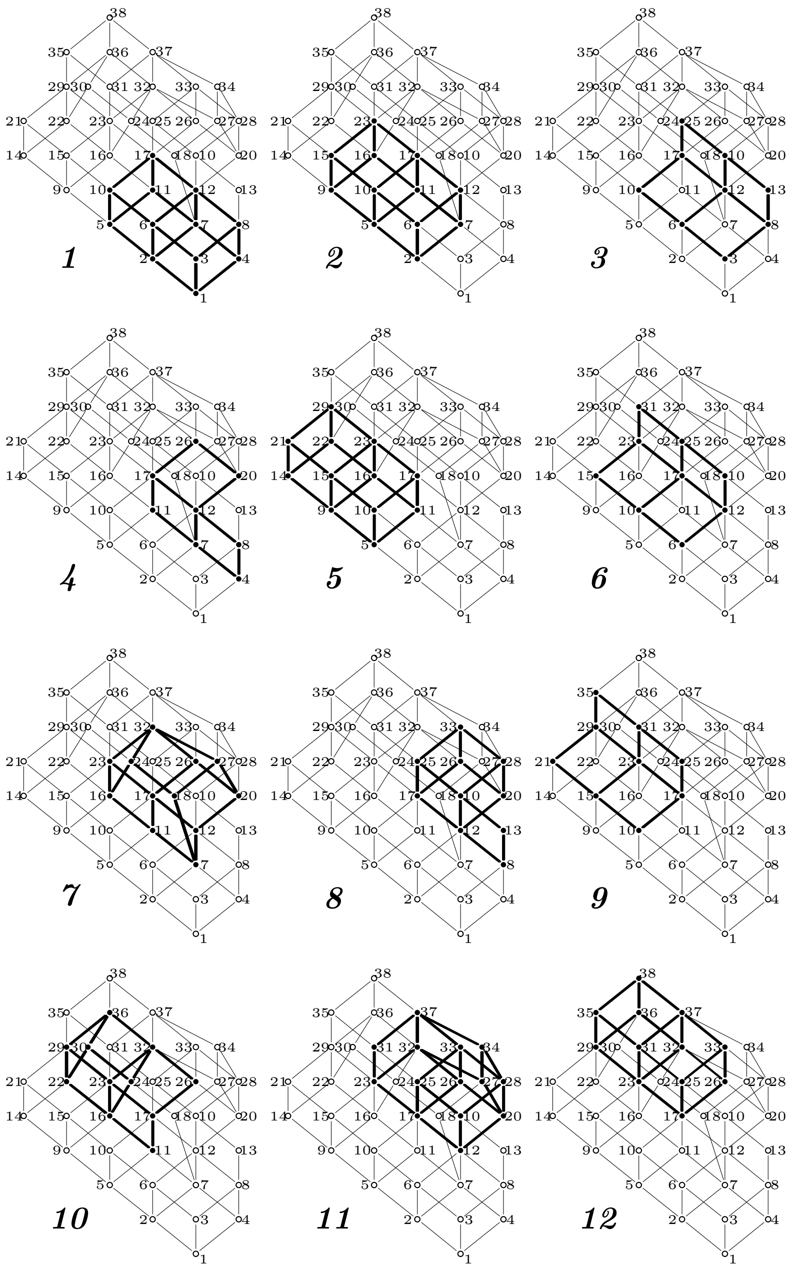

Figure 7.6: Blocks of the tolerance relation $\frac{1}{2}E$ on the concept lattice of Figure 7.3.

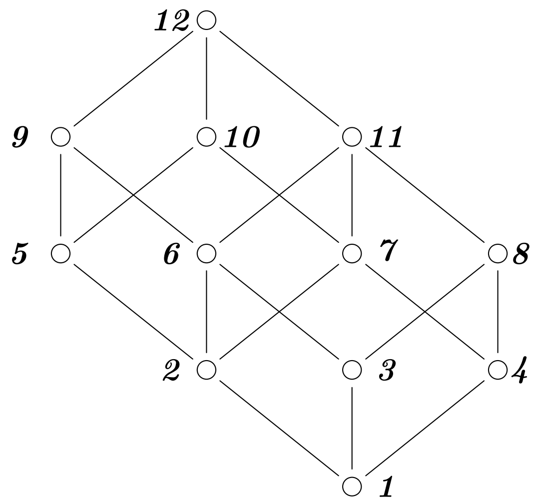

Figure 7.7: Lattice $\mathcal{B}(X, Y, h(I))$. Factor lattice $\mathcal{B}(X, Y, I)/\frac{1}{2}E$.

them empty concepts as well. This is possible if one goes from the original fuzzy concept lattice $\mathcal{B}(X, Y, I)$ to the lattice which results from $\mathcal{B}(X, Y, I)$ by factorization modulo the induced similarity $\frac{1}{2}E$. Indeed, as we can see from Figure 7.3, the concepts between 1 and 17 form one block of $\frac{1}{2}E$ and can thus be considered one concept in the factor lattice. The same applies to other similar concepts of $\mathcal{B}(X, Y, I)$. We thus see that the factorization modulo $\frac{1}{2}E$ makes the concept lattice smaller in that similar concepts which need not be distinguished are put together.

Note that both $\mathcal{B}(X, Y, h(I))$ obtained from $\mathcal{B}(X, Y, I)$ (Gödel structure) and $\mathcal{B}(X, Y, I)/\frac{1}{2}E$ obtained from $\mathcal{B}(X, Y, I)$ (Łukasiewicz structure) are isomorphic. Note also that taking the Gödel structure, $\frac{1}{2}E$ is just the partition induced by the homomorphism h^* induce above, i.e. $\mathcal{B}(X, Y, I)/\frac{1}{2}E$ again is isomorphic to the lattice in Figure 7.7.

Example 7.36 The next example illustrates attribute implications. Table 7.11 shows the input data. There are 17 objects (denoted by 1–17) and 36 attributes (denoted by a–J). In this case, the structure of truth values is the two-element Boolean algebra, i.e. only truth values 0 (false) and 1 (true) are employed. The objects are extinct cephalopods (identified by paleontologists) as listed in Table 7.12. The organisms may posses attributes which are listed in Table 7.13. Because of our purpose we do not comment on the paleontological point of view ((Bělohlávek and Košt'ák, in preparation) contains more information and details).

There is a relatively large number of attributes, obviously more than human mind can grasp at once. Therefore, it is desirable to get information that gives us an additional insight. A suitable one is in the form of attribute dependencies. Table 7.14, Table 7.15, and Table 7.16 show an irredundant basis of all attribute implications that are valid in the input data (that is no one of the implications follows from the others). The implications are to be read as follows. Implications correspond to rows in the table (in order to fit the page, tables are split into two parts: one corresponding to attributes a-r and one corresponding to attributes s-J; therefore, each row is split into two parts). "A" denotes that the corresponding attribute belongs to the antecedent of the implication, "C" denotes that the attribute belongs to the consequent. Thus, the first row says that the implication $\{J\}\Rightarrow\{p\}$ is true in the input data; the eighth row says that the implication $\{A,B\}\Rightarrow\{l\}$ is true in the input data etc.

Computation of concept lattices and attribute implications is too extensive to be done by hand and, therefore, it requires the use of a computer. In our examples we used a software tool that is being developed jointly in the Department of Computer Science, Palacký University, Olomouc (Czech Republic), and in the Department of Computer Science, Technical University of Ostrava (Czech Republic).

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	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r
1	0	0	1	1	0	0	0	0	1	0	0	0	0	1	0	0	1	0
2	1	1	1	1	1	1	1	0	0	1	1	1	0	0	0	1	0	1
3	1	1	1	0	1	1	1	0	0	0	1	1	0	0	0	1	0	1
4	0	1	0	0	1	0	0	0	0	1	0	0	0	0	0	1	0	1
5	1	1	1	0	1	0	0	0	0	0	1	1	0	0	0	1	0	0
6	1	0	0	0	0	1	0	0	0	0	0	1	0	0	0	1	0	0
7	1	1	1	0	1	0	0	0	0	0	1	1	0	0	0	1	1	0
8	0	1	1	0	1	0	0	0	0	0	1	1	0	0	0	1	0	1
9	0	1	0	0	1	0	0	0	0	0	1	1	0	0	0	1	0	1
10	0	1	0	0	0	0	1	0	0	0	0	0	1	0	0	1	0	0
11	0	1	0	0	1	0	0	0	0	0	0	1	0	0	0	1	0	0
12	0	1	0	1	0	0	1	0	0	0	0	1	0	0	0	1	0	0
13	0	1	1	0	0	1	1	0	0	0	0	1	0	0	0	1	0	0
14	0	1	0	0	1	0	0	0	0	0	0	1	0	0	0	1	0	0
15	0	1	1	0	0	0	0	0	0	0	1	1	0	0	0	1	0	0
16	0	0	1	1	0	1	0	0	0	0	0	1	0	0	0	1	0	0
17	0	1	0	0	1	0	0	0	0	0	0	1	0	0	1	1	0	0

	s	t	u	v	w	x	y	z	A	B	C	D	E	F	G	H	I	J
1	0	0	0	0	0	1	0	0	1	0	0	0	0	0	0	0	0	0
2	0	0	1	0	0	0	0	0	1	1	0	0	1	1	0	0	0	1
3	0	0	1	1	0	0	0	0	1	1	0	0	1	1	0	1	1	1
4	0	0	1	1	0	0	0	0	0	1	0	0	0	0	0	0	1	1
5	0	1	0	0	1	0	0	0	1	1	0	0	0	0	0	1	1	1
6	0	1	0	1	0	0	0	0	1	1	0	0	0	1	0	0	0	1
7	0	0	1	0	0	0	0	0	1	1	0	0	1	0	0	0	0	0
8	0	0	1	1	0	0	0	0	1	1	0	0	0	1	0	1	1	1
9	0	0	1	0	0	0	0	0	1	1	0	0	1	0	0	1	1	1
10	0	1	1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	1
11	1	0	1	0	0	0	0	0	1	1	0	0	1	0	0	1	1	0
12	0	1	0	1	0	0	0	0	1	1	0	0	0	0	0	0	0	1
13	0	1	0	1	0	0	0	0	0	1	0	0	0	1	0	0	0	0
14	0	1	1	1	0	0	0	0	0	1	0	0	0	1	0	0	0	1
15	0	1	0	1	0	0	0	0	1	1	0	0	1	0	0	1	1	0
16	1	1	0	1	0	0	0	0	1	1	0	0	0	1	0	1	1	0
17	0	1	1	0	0	0	1	1	1	1	0	0	1	0	0	1	1	1

Table 7.11: Fuzzy context given by fossils and their properties.

1	Actinocamax verus antefragilis
2	Praectinocamax primus
3	Praectinocamax plenus
4	Praectinocamax plenus cf. strehlensis
5	Praectinocamax triangulus
6	Praectinocamax aff triangulus
7	Praectinocamax sozhenzis
8	Praectinocamax contractus
9	Praectinocamax planus
10	Praectinocamax coronatus
11	Praectinocamax matesovae
12	Praectinocamax medwedicicus
13	Praectinocamax sp.1
14	Praectinocamax sp.2
15	Goniocamax intermedius
16	Goniocamax surensis
17	Goniocamax volgensis

Table 7.12: Objects of the context of Table 7.11

a	rostra big
b	rostra medium
c	rostra small
d	cigar shape in dorsoventral view
e	lanceolat in dorsoventral view
f	little lanceolat in dorsoventral view
g	subcylindric in dorsoventral view
h	conic in dorsoventral view
i	cigar shape in lateral view
j	lanceolat in lateral view
k	little lanceolat in lateral view
l	subcylindric in lateral view
m	conic in lateral view
n	flat lateral
o	flat dorsal
p	flat ventral
q	alveolar fracture highly conic
r	alveolar fracture lowly conic
s	pseudoalveol flat
t	pseudoalveol deep
u	cut of alveolar fracture oval
v	cut of alveolar fracture oval-triangular
w	cut of alveolar fracture triangular
x	cut alveolar fracture circle-shaped
y	conellae
z	join
A	dorsolateral line
B	dorsolateral listel
C	rostrum granulation
D	rostrum granulation partly
E	rostrum striation
F	rostrum striation partly
G	vessel engram
H	vessel engram rarely
I	mucro
J	ventral line

Table 7.13: Attributes of the context of Table 7.11

	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r
1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	C	.	.
2	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	C	.	.
3	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
4	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
5	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	C	.	.
6	.	.	.	.	.	.	.	.	.	.	.	A	.	.	.	.	.	.
7	.	.	.	.	.	.	.	.	.	.	.	C	.	.	.	.	.	.
8	.	.	.	.	.	.	.	.	.	.	.	C	.	.	.	.	.	.
9	.	.	.	.	.	.	.	.	.	.	.	C	.	.	.	.	.	.
10	.	.	.	.	.	.	.	.	.	.	.	A	.	.	.	.	.	.
11	.	.	.	.	.	.	.	.	.	.	.	C	.	.	.	.	.	.
12	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
13	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
14	.	.	.	.	.	A	.	.	.	.	.	.	.	.	.	.	.	.
15	.	.	.	.	.	C	.	.	.	.	.	.	.	.	.	.	.	.
16	.	.	.	A	.	.	.	.	.	.	.	.	.	.	.	.	.	.
17	.	.	.	A	.	.	.	.	.	.	.	C	.	.	.	A	.	.
18	.	.	.	A	.	.	.	.	.	.	.	.	.	.	.	.	.	.
19	.	.	.	A	.	.	.	.	.	.	.	.	.	.	.	.	.	.
20	.	.	C	.	.	.	.	.	.	.	.	.	.	.	.	.	A	.
21	.	.	A	.	.	.	.	.	.	.	.	C	.	.	.	A	.	.
22	.	.	C	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
23	.	.	A	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
24	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
25	.	.	A	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
26	.	.	A	.	.	C	.	.	.	.	.	.	.	.	.	.	.	.
27	.	.	.	.	.	.	.	.	C	.	.	.	.	.	.	.	.	.
28	.	.	.	.	.	.	.	.	.	.	.	.	.	A	.	.	.	.
29	.	.	.	C	.	.	.	.	A	.	.	.	.	.	.	.	C	.
30	.	.	.	A	.	.	.	.	.	.	.	.	.	C	.	.	A	.
31	.	.	C	A	.	.	.	.	.	.	.	.	.	.	.	.	.	.
32	.	.	A	A	.	C	.	.	.	.	.	A	.	.	.	.	.	.
33	.	.	C	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
34	.	.	C	A	.	.	.	.	.	.	.	.	.	.	.	.	.	.
35	.	.	A	C	.	.	.	.	.	.	.	.	.	.	.	.	.	.

	s	t	u	v	w	x	y	z	A	B	C	D	E	F	G	H	I	J
1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	A
2	.	.	.	.	.	.	.	.	.	A	.	.	.	.	.	.	.	.
3	.	.	.	.	.	.	.	.	.	C	.	.	.	.	.	.	A	.
4	.	.	.	A	.	.	.	.	.	C	.	.	.	.	.	.	.	.
5	.	A	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
6	.	.	.	.	.	.	.	.	.	C	.	.	.	.	.	.	.	.
7	.	.	.	.	.	.	.	.	.	.	.	.	.	A	.	.	.	.
8	.	.	.	.	.	.	.	.	A	A	.	.	.	.	.	.	.	.
9	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	A	C	.
10	.	.	.	.	.	.	.	.	C	.	.	.	.	.	.	C	A	.
11	.	A	.	.	.	.	.	.	.	A	.	.	.	.	.	.	.	.
12	.	A	.	C	.	.	.	.	.	.	.	.	.	A	.	.	.	.
13	A	.	.	.	.	.	.	.	.	.	.	.	.	.	.	C	.	.
14	.	.	.	.	.	.	.	.	.	.	.	.	.	C	.	.	.	.
15	.	A	.	.	.	.	.	.	A	.	.	.	.	A	.	.	.	.
16	.	.	.	.	.	.	.	.	C	.	.	.	.	.	.	.	.	.
17	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
18	.	C	.	A	.	.	.	.	.	.	.	.	.	.	.	.	.	.
19	.	A	.	C	.	.	.	.	.	.	.	.	.	.	.	.	.	.
20	.	.	.	.	.	.	.	.	C	.	.	.	.	.	.	.	.	.
21	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
22	.	.	.	A	.	.	.	.	.	.	.	.	.	.	.	A	.	.
23	.	.	.	A	.	.	.	.	A	.	.	.	.	.	.	C	.	.
24	.	.	.	C	.	.	.	.	.	.	.	.	.	A	.	A	.	.
25	.	A	.	.	.	.	.	.	A	.	.	.	.	.	.	C	.	.
26	.	A	.	.	.	.	.	.	.	.	.	.	.	A	.	.	.	.
27	.	.	.	.	.	A	.	.	.	.	.	.	.	.	.	.	.	.
28	.	.	.	.	.	C	.	.	.	.	.	.	.	.	.	.	.	.
29	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
30	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
31	.	.	.	.	.	.	.	.	.	.	.	.	.	A	.	.	.	.
32	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
33	A	A	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
34	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	A	.	.
35	A	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.

Table 7.14: Minimal base of implications of the context of Table 7.11 (first part)

	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r
36	.	.	A	.	.	A	.	.	.	.	.	.	.	.	.	.	.	.
37	.	A	.	.	.	.	.	.	.	.	.	.	.	.	.	C	.	.
38	.	C	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
39	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
40	.	C	.	.	.	.	.	.	.	.	.	C	.	.	.	.	.	.
41	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
42	.	C	.	.	.	.	.	.	.	.	A	C	.	.	.	.	.	.
43	.	C	.	.	.	.	A	.	.	.	.	.	.	.	.	.	.	.
44	.	.	.	.	.	.	A	.	.	.	.	.	.	.	.	.	.	.
45	.	.	.	.	.	.	A	.	.	.	.	.	.	.	.	.	.	.
46	.	.	.	.	.	.	A	.	.	.	.	.	.	.	.	.	.	.
47	.	.	.	.	.	.	C	.	.	.	.	.	A	.	.	.	.	.
48	.	.	.	.	.	.	A	.	.	.	.	.	C	.	.	.	.	.
49	.	.	.	.	.	.	A	.	.	.	.	C	.	.	.	.	.	.
50	.	.	.	.	.	.	A	.	.	.	.	A	.	.	.	.	.	.
51	.	C	.	.	A	.	.	.	.	.	.	.	.	.	.	.	.	.
52	.	.	.	.	C	.	.	.	.	.	.	.	.	.	.	.	.	.
53	.	.	.	.	C	.	.	.	.	.	.	.	.	.	.	.	.	.
54	.	.	.	.	A	.	.	.	.	.	.	.	.	.	.	.	.	.
55	.	.	.	.	C	.	.	.	.	.	.	.	.	.	.	.	.	A
56	.	.	.	.	A	.	.	.	.	.	.	.	.	.	.	.	.	C
57	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	A
58	.	A	.	.	C	.	.	.	.	.	.	.	.	.	.	.	.	.
59	.	.	.	.	A	.	.	.	.	.	.	.	.	.	.	.	.	.
60	.	.	.	.	A	.	.	.	.	.	.	.	.	.	.	.	.	.
61	.	.	.	.	C	.	.	.	.	.	.	.	.	.	.	.	.	.
62	.	.	.	.	A	.	.	.	.	.	.	A	.	.	.	.	.	.
63	.	.	.	.	A	.	.	.	.	.	.	.	.	.	.	.	.	.
64	.	.	.	.	A	.	.	.	.	.	.	.	.	.	.	.	.	.
65	.	A	.	.	C	.	.	.	.	.	.	.	.	.	.	.	.	.
66	.	.	.	.	.	.	.	.	.	.	.	.	.	.	C	.	.	.
67	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
68	.	.	.	.	C	.	.	.	.	.	.	.	.	.	A	.	.	.
69	.	.	.	.	A	.	.	.	.	.	.	.	.	.	.	.	.	.
70	.	.	.	.	C	.	.	.	.	.	A	.	.	.	.	.	.	.

	s	t	u	v	w	x	y	z	A	B	C	D	E	F	G	H	I	J
36	C	A	.	.	.	.	.	.	A	.	.	.	.	.	.	.	.	.
37	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
38	.	.	A	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
39	.	A	A	.	.	.	.	.	.	.	.	.	.	.	.	.	.	C
40	.	.	.	.	.	.	.	.	C	.	.	.	A	.	.	.	.	.
41	.	A	.	.	.	.	.	.	.	.	.	.	A	.	.	C	.	.
42	.	.	.	.	.	.	.	.	C	.	.	.	.	.	.	.	.	.
43	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
44	.	.	.	.	.	.	.	.	C	.	.	.	.	.	.	.	.	A
45	.	.	.	.	.	.	.	.	A	.	.	.	.	.	.	.	.	C
46	.	.	A	.	.	.	.	.	C	.	.	.	.	.	.	.	.	.
47	.	C	C	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
48	.	A	A	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
49	.	.	.	.	.	.	.	.	.	A	.	.	.	.	.	.	.	.
50	.	A	.	C	.	.	.	.	.	.	.	.	.	.	.	.	.	.
51	.	.	.	.	.	.	.	.	.	C	.	.	.	.	.	.	.	.
52	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	A	A
53	.	.	A	.	.	.	.	.	.	A	.	.	.	.	.	.	.	.
54	.	.	C	A	.	.	.	.	.	.	.	.	.	.	.	.	.	C
55	.	.	C	.	.	.	.	.	.	.	.	.	.	.	.	.	.	C
56	.	.	.	A	.	.	.	.	.	.	.	.	.	.	.	.	A	.
57	.	.	.	A	.	.	.	.	.	.	.	.	.	.	.	.	C	.
58	.	.	.	.	.	.	.	.	.	.	.	.	.	A	.	.	.	A
59	.	.	C	.	.	.	.	.	.	.	.	.	.	A	.	.	.	C
60	.	.	C	.	.	.	.	.	.	.	.	.	A	.	.	.	.	.
61	.	.	.	.	.	.	.	.	.	.	.	.	A	.	.	.	.	A
62	.	.	.	A	.	.	.	.	.	.	.	.	.	C	.	.	.	.
63	.	A	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	C
64	.	A	.	.	.	.	.	.	A	.	.	.	.	.	.	C	.	.
65	A	.	.	.	.	.	.	.	.	.	.	.	C	.	.	.	.	.
66	.	.	.	.	.	.	.	A	.	.	.	.	.	.	.	.	.	.
67	.	.	.	.	.	.	A	C	.	.	.	.	.	.	.	.	.	.
68	.	C	.	.	.	.	.	.	.	.	.	.	C	.	.	.	.	.
69	.	A	A	.	.	.	C	.	A	.	.	.	.	.	.	.	.	.
70	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	A

Table 7.15: Minimal base of implications of the context of Table 7.11 (second part)

	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r
71	.	.	.	.	A	.	.	.	.	.	A	.	.	.	.	.	.	.
72	.	.	.	.	.	.	.	.	.	.	C	A	.	.	.	.	.	A
73	.	.	.	.	.	.	.	.	.	.	A	.	.	.	.	.	.	C
74	.	.	.	.	.	.	.	.	.	A	.	.	.	.	.	.	.	C
75	.	.	.	.	.	.	.	.	.	A	.	.	.	.	.	.	.	.
76	.	C	.	A	.	.	.	.	.	.	.	.	.	.	.	.	.	.
77	.	A	.	A	.	.	C	.	.	.	.	.	.	.	.	.	.	.
78	.	A	.	C	.	.	.	.	.	.	.	.	.	.	.	.	.	.
79	.	A	A	.	.	.	.	.	.	.	C	.	.	.	.	.	.	.
80	.	.	C	.	.	.	.	.	.	.	A	.	.	.	.	.	.	.
81	.	.	C	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
82	.	.	A	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
83	.	.	C	.	.	.	.	.	.	.	A	.	.	.	.	.	.	.
84	.	.	.	.	.	.	.	.	.	.	A	.	.	.	.	.	.	.
85	.	.	C	.	.	.	A	.	.	.	.	.	.	.	.	.	.	.
86	.	A	.	.	.	A	C	.	.	.	.	.	.	.	.	.	.	.
87	.	.	A	.	.	C	A	.	.	.	.	.	.	.	.	.	.	.
88	.	.	A	.	A	.	.	.	.	.	C	.	.	.	.	.	.	.
89	.	.	A	.	C	.	.	.	.	.	.	.	.	.	.	.	.	.
90	.	A	C	.	C	.	.	.	.	.	.	.	.	.	.	.	.	.
91	.	.	A	.	.	.	.	.	.	.	.	.	.	.	.	.	.	A
92	A	.	.	.	.	.	.	.	.	.	.	C	.	.	.	.	.	.
93	A	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
94	C	.	.	.	.	A	.	.	.	.	.	.	.	.	.	.	.	.
95	A	.	.	.	.	C	.	.	.	.	.	.	.	.	.	.	.	.
96	A	.	.	.	.	C	.	.	.	.	.	.	.	.	.	.	.	.
97	A	C	A	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
98	A	A	C	.	C	.	.	.	.	.	.	.	.	.	.	.	.	.
99	A	C	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
100	C	.	A	.	A	.	.	.	.	.	.	.	.	.	.	.	.	.
101	A	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
102	C	.	C	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
103	.	.	A	.	A	.	.	.	.	.	.	.	.	.	.	.	.	.
104	C	.	.	.	.	.	.	.	.	.	.	A	.	.	.	.	A	.
105	C	.	.	.	A	.	A	.	.	.	.	.	.	.	.	.	.	.
106	.	.	.	C	.	.	.	.	.	A	A	.	.	.	.	.	.	.
107	A	.	.	A	.	.	.	.	.	C	.	.	.	.	.	.	.	.
108	.	.	.	.	.	.	.	C	.	.	.	.	.	.	.	.	.	.
109	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
110	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
111	.	.	.	.	.	C	.	A	C	.	.	.	.	.	.	.	.	.
112	.	.	.	.	.	.	.	C	.	.	.	.	.	.	.	.	.	.
113	.	.	.	.	.	A	.	.	.	.	.	.	C	.	.	.	A	.

	s	t	u	v	w	x	y	z	A	B	C	D	E	F	G	H	I	J
71	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	A	.	C
72	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
73	.	.	A	.	.	.	.	.	.	.	.	.	.	.	.	.	.	A
74	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
75	.	.	.	C	.	.	.	.	.	.	.	.	.	.	.	.	A	.
76	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	A
77	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
78	.	A	.	A	.	.	.	.	A	.	.	.	.	.	.	.	.	A
79	.	.	.	.	.	.	.	.	A	.	.	.	.	.	.	.	.	.
80	.	.	.	A	.	.	.	.	.	.	.	.	.	.	.	.	.	.
81	.	.	.	A	.	.	.	.	.	.	.	.	A	.	.	.	.	.
82	.	.	.	C	.	.	.	.	.	.	.	.	A	.	.	A	.	.
83	.	A	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
84	.	A	.	A	.	.	.	.	.	.	.	.	C	.	.	.	.	.
85	.	.	.	.	.	.	.	.	.	.	.	.	.	A	.	.	.	.
86	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
87	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
88	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
89	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	A
90	.	.	.	.	.	.	.	.	A	.	.	.	.	A	.	.	.	.
91	.	.	.	.	.	.	.	.	.	.	.	.	.	C	.	.	.	.
92	.	.	.	.	.	.	.	.	C	.	.	.	.	.	.	.	.	.
93	.	A	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	C
94	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	A
95	.	.	.	.	.	.	.	.	.	.	.	.	.	A	.	.	.	C
96	.	.	.	A	.	.	.	.	.	.	.	.	.	.	.	.	.	.
97	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
98	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
99	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	A	.	.
100	.	.	.	.	.	.	.	.	.	.	.	.	A	.	.	.	.	.
101	.	.	A	.	.	.	.	.	.	.	.	.	C	.	.	.	.	.
102	.	C	.	.	A	.	.	.	.	.	.	.	.	.	.	.	.	.
103	.	A	.	.	C	.	.	.	.	.	.	.	.	.	.	.	.	.
104	.	.	C	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
105	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.

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