

## Fuzzy Closure Operators

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### 1. INTRODUCTION

Closure operators (and closure systems) play a significant role in both pure and applied mathematics. In the framework of fuzzy set theory, several particular examples of closure operators and systems have been considered (e.g. so-called fuzzy subalgebras, fuzzy congruences, fuzzy topology etc.). Recently, fuzzy closure operators and fuzzy closure systems themselves (i.e. operators which map fuzzy sets to fuzzy sets and the corresponding systems of closed fuzzy sets) have been studied by Gerla et al., see e.g. [3, 4, 6, 7]. As a matter of fact, a fuzzy set  $A$  is usually defined as a mapping from a universe set  $X$  into the real interval  $[0, 1]$  in the above mentioned works. Therefore, the structure of truth values of the “logic behind” is fixed to  $[0, 1]$  equipped usually with minimum as the operation corresponding to logical conjunction.

As it appeared recently in the investigations of fuzzy logic in narrow sense [9, 10] (i.e. fuzzy logic as a many-valued logical calculus), there are several logical calculi formalizing the intuitive idea of “fuzzy reasoning” which are complete with respect to the semantics over special structures of truth values. Among these structures, the most general one is that of a residuated lattice (it is worth noticing that residuated lattices (introduced originally in [12] as an abstraction in the study of ideal systems of rings) have been proposed as a suitable structure of truth values by Goguen in [8]). From this point of view, the need for a general notion of a “fuzzy closure” concept becomes apparent.

The aim of this paper is to outline a general theory of fuzzy closure operators and fuzzy closure systems. In the next section we introduce the necessary concepts. In Section 3, fuzzy closure operators and systems are

defined and investigated. The notions defined here generalize the notions introduced and studied earlier in two directions: First, as indicated above, arbitrary (in fact, complete) residuated lattice is used for the structure of truth values (as a matter of fact,  $[0, 1]$  equipped by minimum is itself a complete residuated lattice, cf. Section 2). Second, we generalize the usual monotonicity condition so that it reads “if  $A$  is *almost* a subset of  $B$  then the closure of  $A$  is *almost* a subset of the closure of  $B$ ”. Investigated is also the relationship to fuzzy Galois connections.

## 2. PRELIMINARIES

A fuzzy set in a universe set  $X$  is any mapping from  $X$  into  $L$ ,  $L$  being an appropriate set of truth values.  $L$  has to be equipped with some structure. A general one is that of a complete residuated lattice.

DEFINITION 2.1. A *complete residuated lattice* is an algebra  $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1 \rangle$  such that

- (1)  $\langle L, \wedge, \vee, 0, 1 \rangle$  is a complete lattice with the least element 0 and the greatest element 1,
- (2)  $\langle L, \otimes, 1 \rangle$  is a commutative monoid, i.e.  $\otimes$  is commutative, associative, and  $x \otimes 1 = x$  holds for each  $x \in L$ , and
- (3)  $\otimes, \rightarrow$  form an adjoint pair, i.e.

$$x \otimes y \leq z \text{ iff } x \leq y \rightarrow z \quad (1)$$

holds for all  $x, y, z \in L$ .

In each residuated lattice it holds that  $x \leq y$  implies  $x \otimes z \leq y \otimes z$  (isotonicity), and  $x \leq y$  implies  $z \rightarrow x \leq z \rightarrow y$  (isotonicity in the second argument) and  $x \rightarrow z \geq y \rightarrow z$  (antitonicity in the first argument).  $\otimes$  and  $\rightarrow$  are called *multiplication* and *residuum*, respectively.

The most studied and applied set of truth values is the real interval  $[0, 1]$  with  $a \wedge b = \min(a, b)$ ,  $a \vee b = \max(a, b)$ , and with three important pairs of adjoint operations: the Łukasiewicz one ( $a \otimes b = \max(a + b - 1, 0)$ ,  $a \rightarrow b = \min(1 - a + b, 1)$ ), Gödel one ( $a \otimes b = \min(a, b)$ ,  $a \rightarrow b = 1$  if  $a \leq b$  and  $= b/a$  else), and product one ( $a \otimes b = a \cdot b$ ,  $a \rightarrow b = 1$  if  $a \leq b$  and  $= b/a$  else). For the role of these “building stones” in fuzzy logic see [9]. Another important set of truth values is the set  $\{a_0 = 0, a_1, \dots, a_n = 1\}$  ( $a_0 < \dots < a_n$ ) with  $\otimes$  given by  $a_k \otimes a_l = a_{\max(k+l-n, 0)}$  and the corresponding  $\rightarrow$  given by  $a_k \rightarrow a_l = a_{\min(n-k+l, n)}$ . A special case of the latter algebras is the Boolean algebra  $\mathbf{2}$  of classical logic with the support  $2 = \{0, 1\}$ . It may be easily verified that the only multiplication on  $\{0, 1\}$  is the classical

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conjunction operation  $\wedge$ , i.e.  $a \wedge b = 1$  iff  $a = 1$  and  $b = 1$ , which implies that the only residuum operation is the classical implication operation  $\rightarrow$ , i.e.  $a \rightarrow b = 0$  iff  $a = 1$  and  $b = 0$ . Note that each of the preceding residuated lattices is complete.

Multiplication  $\otimes$  and residuum  $\rightarrow$  are intended for modeling of the conjunction and implication, respectively. Supremum ( $\vee$ ) and infimum ( $\wedge$ ) are intended for modeling of general and existential quantifier, respectively. A syntactico-semantically complete first-order logic with truth values in complete residuated lattices can be found in [10].

A nonempty subset  $K \subseteq L$  is called an  $\leq$ -filter if for every  $a, b \in L$  such that  $a \leq b$  it holds that  $b \in K$  whenever  $a \in K$ . An  $\leq$ -filter  $K$  is called a filter if  $a, b \in K$  implies  $a \otimes b \in K$ . Unless otherwise stated, in what follows we denote by  $\mathbf{L}$  a complete residuated lattice and by  $K$  an  $\leq$ -filter in  $\mathbf{L}$  (both  $\mathbf{L}$  and  $K$  possibly with indices).

An  $\mathbf{L}$ -set (fuzzy set) [13, 8]  $A$  in a universe set  $X$  is any map  $A : X \rightarrow L$ . By  $L^X$  we denote the set of all  $\mathbf{L}$ -sets in  $X$ . The concept of an  $\mathbf{L}$ -relation is defined obviously. Operations on  $L$  extend pointwise to  $L^X$ , e.g.  $(A \vee B)(x) = A(x) \vee B(x)$  for  $A, B \in L^X$ . Following common usage, we write  $A \cup B$  instead of  $A \vee B$ , etc. Given  $A, B \in L^X$ , the *subsethood degree* [8]  $S(A, B)$  of  $A$  in  $B$  is defined by  $S(A, B) = \bigwedge_{x \in X} A(x) \rightarrow B(x)$ . We write  $A \subseteq B$  if  $S(A, B) = 1$ . Analogously, the *equality degree*  $E(A, B)$  of  $A$  and  $B$  is defined by  $E(A, B) = \bigwedge_{x \in X} (A(x) \leftrightarrow B(x))$ . It is immediate that  $E(A, B) = S(A, B) \wedge S(B, A)$ .

### 3. $\mathbf{L}_K$ -CLOSURE OPERATORS AND $\mathbf{L}_K$ -CLOSURE SYSTEMS

Recall that an closure operator on a set  $X$  is a mapping  $C : 2^X \rightarrow 2^X$  satisfying the following conditions:  $A \subseteq C(A)$ , if  $A_1 \subseteq A_2$  then  $C(A_1) \subseteq C(A_2)$ , and  $C(A) = C(C(A))$ , for any  $A, A_1, A_2 \in 2^X$ . More generally, if  $\subseteq$  denotes a partial order, we get the notion of closure operator in an ordered set [5].

DEFINITION 3.1. An  $\mathbf{L}_K$ -closure operator (fuzzy closure operator) on the set  $X$  is a mapping  $C : L^X \rightarrow L^X$  satisfying

$$A \subseteq C(A) \tag{2}$$

$$S(A_1, A_2) \leq S(C(A_1), C(A_2)) \quad \text{whenever } S(A_1, A_2) \in K \tag{3}$$

$$C(A) = C(C(A)) \tag{4}$$

for every  $A, A_1, A_2 \in L^X$ .

If  $K = L$ , we omit the subscript  $K$  and call  $C$  an  $\mathbf{L}$ -closure operator. The set  $K$  plays the role of the set of designated truth values. Condition (3) says that the closure preserves also partial subsethood whenever the subsethood degree is designated. Since  $K$  is an  $\leq$ -filter in  $\mathbf{L}$ , the designated truth values represent, in a sense, sufficiently high truth values. In this view, (3) reads “if  $A_1$  is almost included in  $A_2$  then  $C(A_1)$  is almost included in  $C(A_2)$ ”. It is easily seen that each  $\mathbf{L}_K$ -closure operator on  $X$  is an closure operator on the complete lattice  $\langle L^X, \subseteq \rangle$  [5].

*Remark 3. 1.* Note that for  $L = \{0, 1\}$ ,  $\mathbf{L}_K$ -closure operators are precisely the classical closure operators. Clearly, if  $K_1 \subseteq K_2$  then each  $\mathbf{L}_{K_2}$ -closure operator is an  $\mathbf{L}_{K_1}$ -closure operator. As we will see, the converse is not true. Note also that for  $L = [0, 1]$ ,  $\mathbf{L}_{\{1\}}$ -closure operators are precisely fuzzy closure operators studied by Gerla [4, 6, 7].

*Remark 3. 2.* We show that for residuated lattices  $\mathbf{L}$  with  $L = [0, 1]$  of Lukasiewicz, Gödel, and product logic [9], the set  $K$  is relevant: Take  $X = \{x_1, x_2\}$ , and define  $C$  by  $C(A)(x_1) = 0, C(A)(x_2) = 0.5$  for  $A(x_1) = 0, A(x_2) \leq 0.5$ , and  $C(A)(x_1) = C(A)(x_2) = 1$  otherwise. An easy inspection shows that  $C$  is an  $\mathbf{L}_{\{1\}}$ -closure operator. However, for  $A_1, A_2$  given by  $A_1(x_1) = A_2(x_1) = 0, A_1(x_2) = 1, A_2(x_2) = 0.5$  it holds  $S(A_1, A_2) = 0.5 > 0 = S(C(A_1), C(A_2))$  (for all of the three algebras). Thus,  $C$  is not an  $\mathbf{L}_{[0.5, 1]}$ -closure operator.

**THEOREM 3.1.**  $C : L^X \rightarrow L^X$  is an  $\mathbf{L}_K$ -closure operator on  $X$  iff it satisfies (2) and the following condition

$$S(A_1, C(A_2)) \leq S(C(A_1), C(A_2)) \quad \text{whenever } S(A_1, C(A_2)) \in K. \quad (5)$$

*Proof.* Suppose (2)–(4) hold. If  $S(A_1, C(A_2)) \in K$  then by (3) and (4) we have  $S(A_1, C(A_2)) \leq S(C(A_1), C(C(A_2))) = S(C(A_1), C(A_2))$ , i.e. (5) holds.

Conversely, let (2) and (5) hold. Suppose  $S(A_1, A_2) \in K$ . Since, by (2),  $A_2 \subseteq C(A_2)$ , we have  $S(A_1, A_2) \leq S(A_1, C(A_2)) \in K$ . Furthermore, (5) implies  $S(A_1, C(A_2)) \leq S(C(A_1), C(A_2))$ , hence  $S(A_1, A_2) \leq S(C(A_1), C(A_2))$ , proving (3). By (5),  $1 = S(C(A), C(A)) \leq S(C(C(A)), C(A))$ , we have  $C(C(A)) \subseteq C(A)$ . Since the converse inclusion holds by (2), we conclude (4). ■

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DEFINITION 3.2. A system  $\mathcal{S} = \{A_i \in L^X \mid i \in I\}$  is called *closed under  $S_K$ -intersections* iff for each  $A \in L^X$  it holds that

$$\bigcap_{i \in I, S(A, A_i) \in K} S(A, A_i) \rightarrow A_i \in \mathcal{S}$$

where

$$\left( \bigcap_{i \in I, S(A, A_i) \in K} S(A, A_i) \rightarrow A_i \right)(x) = \bigwedge_{i \in I, S(A, A_i) \in K} (S(A, A_i) \rightarrow A_i(x))$$

for each  $x \in X$ . A system closed under  $S_K$ -intersections will be called an  $L_K$ -closure system.

For  $K = L$  the subscript will again be omitted.

*Remark 3.3.* (1) We have

$$\bigcap_{i \in I, S(A, A_i) \in \{1\}} S(A, A_i) \rightarrow A_i = \bigcap_{i \in I, A \subseteq A_i} A_i.$$

Therefore,  $\mathcal{S}$  is a **2**-closure system iff for each  $A \subseteq X$  it holds  $\bigcap_{A \subseteq A_i} A_i \in \mathcal{S}$ . It can be easily seen that the last condition is equivalent to being closed under arbitrary intersection for  $\mathcal{S}$ . Hence, **2**-closure systems coincide with closure systems, i.e. systems of sets closed under intersections [5].

(2) In general, being closed under arbitrary intersections is a weaker condition than being closed under  $S_K$ -intersections. Indeed, let  $\mathcal{S}$  be closed under  $S_K$ -intersections. To show that  $\mathcal{S}$  is closed under arbitrary intersections, it suffices to show that

$$\bigwedge_{j \in J} A_j(x) = \bigwedge_{i \in I, S(\bigcap_{j \in J} A_j, A_i) \in K} (S(\bigcap_{j \in J} A_j, A_i) \rightarrow A_i(x))$$

holds for any  $J \subseteq I$ . The inequality  $\geq$  is clearly valid since for each  $j' \in J$  we have  $S(\bigcap_{j \in J} A_j, A_{j'}) \rightarrow A_{j'}(x) = 1 \rightarrow A_{j'}(x) = A_{j'}(x)$ . The converse inequality holds iff

$$\bigwedge_{j \in J} A_j(x) \leq S(\bigcap_{j \in J} A_j, A_i) \rightarrow A_i(x)$$

for each  $i \in I$  such that  $S(\bigcap_{j \in J} A_j, A_i)$  which is equivalent to

$$\bigwedge_{j \in J} A_j(x) \otimes S(\bigcap_{j \in J} A_j, A_i) \leq A_i(x),$$

i.e.

$$\bigwedge_{j \in J} A_j(x) \otimes \bigwedge_{y \in X} (\bigwedge_{j \in J} A_j(y)) \rightarrow A_i(y) \leq A_i(x)$$

which holds because

$$\begin{aligned} & \bigwedge_{j \in J} A_j(x) \otimes \bigwedge_{y \in X} (\bigwedge_{j \in J} A_j(y)) \rightarrow A_i(y) \leq \\ & \leq \bigwedge_{j \in J} A_j(x) \otimes (\bigwedge_{j \in J} A_j(x)) \rightarrow A_i(x) \leq A_i(x). \end{aligned}$$

On the other hand, put  $X = \{x\}$ , take the Łukasiewicz structure with  $\mathbf{L} = \{0, \frac{1}{2}, 1\}$ ,  $K = L$ ,  $\mathcal{S} = \{\{0/x\}, \{1/x\}\}$ , and  $A = \{\frac{1}{2}/x\}$ . Then  $\mathcal{S}$  is clearly closed under arbitrary intersections but not under  $S_K$ -intersections since  $\bigcap_{i \in I, S(A, A_i) \in K} S(A, A_i) \rightarrow A_i = A \notin \mathcal{S}$ .

Closedness under  $S_K$ -intersections is, however, equivalent to closedness under intersections of “ $K$ -shifted”  $\mathbf{L}$ -sets. Let for  $a \in L$ ,  $A \in L^X$ , denote by  $a \rightarrow A$  the  $\mathbf{L}$ -set defined by  $(a \rightarrow A)(x) = a \rightarrow A(x)$ .

**THEOREM 3.2.**  *$\mathcal{S}$  is an  $\mathbf{L}_K$ -closure system iff for any  $a_i \in L$ ,  $i \in I$ , it holds  $\bigcap_{a_i \in K} (a_i \rightarrow A_i) \in \mathcal{S}$ .*

*Proof.* If  $\bigcap_{a_i \in K} (a_i \rightarrow A_i) \in \mathcal{S}$  for every  $a_i \in L$  then taking  $a_i = S(A, A_i)$  if  $S(A, A_i) \in K$  and  $a_i = 0$  otherwise for  $A \in L^X$ , it is easily seen that  $\bigwedge_{i \in I, S(A, A_i) \in K} S(A, A_i) \rightarrow A_i(x) = \bigwedge_{a_i \in K} (a_i \rightarrow A_i(x))$ , i.e.  $\mathcal{S}$  is an  $\mathbf{L}_K$ -closure system.

Conversely, let  $\mathcal{S}$  be an  $\mathbf{L}_K$ -closure system. Let  $a_i \in L$  and put  $A = \bigcap_{a_i \in K} (a_i \rightarrow A_i)$ . We have to show  $A \in \mathcal{S}$ . Clearly, it is enough to show that  $\bigcap_{i \in I, S(A, A_i) \in K} (S(A, A_i) \rightarrow A_i) = A$ . The fact

$$\bigcap_{i \in I, S(A, A_i) \in K} (S(A, A_i) \rightarrow A_i) \supseteq A$$

is shown in Lemma 3.1. For the converse inclusion, observe first that if  $a_j \in K$  then  $S(A, A_j) \in K$ . Indeed, by  $\leq$ -filter property of  $K$  it suffices to show that  $a_j \leq S(A, A_j)$ . This holds iff for each  $x \in X$  it holds

$$a_j \leq (\bigwedge_{a_i \in K} (a_i \rightarrow A_i(x)) \rightarrow A_j(x))$$

, i.e.  $a_j \otimes (\bigwedge_{a_i \in K} (a_i \rightarrow A_i(x)) \leq A_j(x))$  which holds since  $a_j \otimes (a_j \rightarrow A_j(x)) \leq A_j(x)$ . Now, the converse inclusion holds iff for each  $x \in X$  we

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have

$$\bigwedge_{i \in I, S(A, A_i) \in K} S(A, A_i) \rightarrow A_i(x) \leq \bigwedge_{a_i \in K} a_i \rightarrow A_i(x)$$

which holds iff for each  $a_j \in K$  it holds

$$a_j \otimes \bigwedge_{i \in I, S(A, A_i) \in K} S(A, A_i) \rightarrow A_i(x) \leq A_j(x)$$

which holds since by the above observation  $S(A, A_j) \in K$ , and therefore

$$a_j \otimes \bigwedge_{i \in I, S(A, A_i) \in K} S(A, A_i) \rightarrow A_i(x) \leq a_j \otimes (S(A, A_j) \rightarrow A_j(x)) \leq A_j(x).$$

The theorem is proved. ■

**COROLLARY 3.1.** *A system  $\mathcal{S}$  which is closed under arbitrary intersections is an  $\mathbf{L}_K$ -closure system iff for each  $a \in K$  and  $A \in \mathcal{S}$  it holds  $a \rightarrow A \in \mathcal{S}$ .*

The following theorem shows another way to obtain the closure in an  $\mathbf{L}_K$ -closure system.

**THEOREM 3.3.** *Let  $\mathcal{S} = \{A_i \in L^X \mid i \in I\}$  be an  $\mathbf{L}_K$ -closure system. Then for each  $A \in L^X$  it holds*

$$\bigcap_{i \in I, S(A, A_i) \in K} S(A, A_i) \rightarrow A_i = \bigcap_{i \in I, A \subseteq A_i} A_i.$$

*Proof.* Clearly,

$$\bigcap_{i \in I, S(A, A_i) \in K} S(A, A_i) \rightarrow A_i \subseteq \bigcap_{i \in I, S(A, A_i) = 1} S(A, A_i) \rightarrow A_i = \bigcap_{i \in I, A \subseteq A_i} A_i.$$

On the other hand, it is easy to check that  $A \subseteq \bigcap_{i \in I, S(A, A_i) \in K} S(A, A_i) \rightarrow A_i$ . Since  $\mathcal{S}$  is an  $\mathbf{L}_K$ -closure system, we have  $\bigcap_{i \in I, S(A, A_i) \in K} S(A, A_i) \rightarrow A_i \in \mathcal{S}$  which immediately gives  $\bigcap_{i \in I, A \subseteq A_i} A_i \subseteq \bigcap_{i \in I, S(A, A_i) \in K} S(A, A_i) \rightarrow A_i$ . ■

**LEMMA 3.1.** *Let  $\mathcal{S} = \{A_i \mid i \in I\}$  be an  $\mathbf{L}_K$ -closure system,  $K$  be a filter in  $\mathbf{L}$ . Then  $C_S : L^X \rightarrow L^X$  defined by  $C_S(A)(x) = \bigwedge_{i \in I, S(A, A_i) \in K} (S(A, A_i) \rightarrow A_i(x))$  is an  $\mathbf{L}_K$ -closure operator. Moreover, for  $A \in L^X$  it holds  $A \in \mathcal{S}$  iff  $A = C_S(A)$ .*

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*Proof.* We check (2)–(4).

(2): We have to show  $A(x) \leq C_S(A)(x)$  for each  $x \in X$  which holds iff for each  $i \in I$  such that  $S(A, A_i) \in K$  it holds  $A(x) \leq S(A, A_i) \rightarrow A_i(x)$ . This is, by adjointness, equivalent to  $A(x) \otimes S(A, A_i) \leq A_i(x)$ , i.e.  $A(x) \otimes \bigwedge_{y \in X} (A(y) \rightarrow A_i(y)) \leq A_i(x)$  which holds because of

$$A(x) \otimes \bigwedge_{y \in X} (A(y) \rightarrow A_i(y)) \leq A(x) \otimes (A(x) \rightarrow A_i(x)) \leq A_i(x).$$

(3): Suppose  $S(A_1, A_2) \in K$ . We have to show

$$S(A_1, A_2) \leq S(C_S(A_1), C_S(A_2))$$

which is equivalent to the fact that for each  $x \in X$  it holds  $S(A_1, A_2) \leq C_S(A_1)(x) \rightarrow C_S(A_2)(x)$ , i.e. by adjointness,

$$C_S(A_1)(x) \otimes S(A_1, A_2) \leq C_S(A_2)(x) = \bigwedge_{i \in I, S(A_2, A_i) \in K} S(A_2, A_i) \rightarrow A_i(x)$$

which is true iff for each  $j \in I$  with  $S(A_2, A_j) \in K$  it holds

$$C_S(A_1)(x) \otimes S(A_1, A_2) \otimes S(A_2, A_j) \leq A_j(x)$$

which is true. Indeed, since

$$S(A_1, A_2) \otimes S(A_2, A_j) \leq S(A_1, A_j),$$

$S(A_1, A_2) \in K$  and  $S(A_2, A_j) \in K$ , the filter property of  $K$  yields  $S(A_1, A_j) \in K$ , and we have

$$\begin{aligned} C_S(A_1)(x) \otimes S(A_1, A_2) \otimes S(A_2, A_j) &= \\ &= S(A_1, A_2) \otimes S(A_2, A_j) \otimes \bigwedge_{i \in I, S(A_1, A_i) \in K} (S(A_1, A_i) \rightarrow A_i(x)) \leq \\ &\leq S(A_1, A_j) \otimes (S(A_1, A_j) \rightarrow A_j(x)) \leq A_j(x). \end{aligned}$$

(4): Clearly, we only have to show  $C_S(C_S(A)) \subseteq C_S(A)$ . Since  $C_S(A) \in \mathcal{S}$ , there is some  $j \in I$  such that  $A_j = C_S(A)$ . We therefore have

$$\begin{aligned} C_S(C_S(A))(x) &= \bigwedge_{j \in I, S(C_S(A), A_j) \in K} S(C_S(A), A_j) \rightarrow A_j(x) \leq \\ &\leq S(C_S(A), C_S(A)) \rightarrow (C_S(A))(x) = (C_S(A))(x). \end{aligned}$$

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We now show that  $A \in \mathcal{S}$  iff  $A = C_{\mathcal{S}}(A)$ . Indeed, if  $A = A_j \in \mathcal{S}$  then  $A_j \subseteq C_{\mathcal{S}}(A_j)$  as proved above.

Conversely,

$$C_{\mathcal{S}}(A_j)(x) = \bigwedge_{i \in I, S(A_j, A_i) \in K} (S(A_j, A_i) \rightarrow A_i(x)) \leq S(A_j, A_j) \rightarrow A_j(x) \leq A_j(x),$$

i.e.  $C_{\mathcal{S}}(A_j) \subseteq A_j$ . If  $A = C_{\mathcal{S}}(A)$  then  $A \in \mathcal{S}$  by the definition of the  $\mathbf{L}_K$ -closure system, completing the proof. ■

LEMMA 3.2. *Let  $C : L^X \rightarrow L^X$  be an  $\mathbf{L}_K$ -closure operator. Then  $\mathcal{S}_C = \{A \in L^X \mid A = C(A)\}$  is an  $\mathbf{L}_K$ -closure system.*

*Proof.* Let  $I$  be such that  $\mathcal{S}_C = \{A_i \mid i \in I\}$ . We have to show that for each  $A \in L^X$  it holds  $\bigwedge_{i \in I, S(A, A_i) \in K} (S(A, A_i) \rightarrow A_i) \in \mathcal{S}_C$ . To this end it clearly suffices to show

$$\bigwedge_{i \in I, S(A, A_i) \in K} (S(A, A_i) \rightarrow A_i) = C(A). \quad (6)$$

On the one hand, since  $S(A, C(A)) = 1 \in K$ , we have

$$\bigwedge_{i \in I, S(A, A_i) \in K} (S(A, A_i) \rightarrow A_i(x)) \leq S(A, C(A)) \rightarrow C(A)(x) = C(A)(x).$$

On the other hand,

$$C(A)(x) \leq \bigwedge_{i \in I, S(A, A_i) \in K} (S(A, A_i) \rightarrow A_i(x))$$

iff for each  $i \in I$  such that  $S(A, A_i) \in K$  it holds  $C(A)(x) \otimes S(A, A_i) \leq A_i(x)$ . This is, indeed, true since

$$\begin{aligned} C(A)(x) \otimes S(A, A_i) &\leq C(A)(x) \otimes S(C(A), C(A_i)) \leq \\ &\leq C(A)(x) \otimes (C(A)(x) \rightarrow C(A_i)(x)) \leq C(A_i)(x) = A_i(x). \end{aligned}$$

To sum up, (6) is proved. ■

THEOREM 3.4. *Let  $C$  be an  $\mathbf{L}_K$ -closure operator on  $X$ ,  $\mathcal{S}$  be an  $\mathbf{L}_K$ -closure system on  $X$ ,  $K$  be a filter in  $\mathbf{L}$ . Then  $\mathcal{S}_C$  is an  $\mathbf{L}_K$ -closure system on  $X$ ,  $C_{\mathcal{S}}$  is an  $\mathbf{L}_K$ -closure operator on  $X$  and it holds  $C = C_{\mathcal{S}_C}$  and  $\mathcal{S} = \mathcal{S}_{C_{\mathcal{S}}}$ , i.e. the mappings  $C \mapsto \mathcal{S}_C$  and  $\mathcal{S} \mapsto C_{\mathcal{S}}$  are mutually inverse.*

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*Proof.* By Lemma 3.1 and Lemma 3.2 it remains to prove  $C = C_{S_C}$ , i.e. that for any  $A \in L^X$ ,  $x \in X$ , it holds

$$C(A)(x) = \bigwedge_{A' \in L^X, A' = C(A'), S(A, A') \in K} (S(A, A') \rightarrow A'(x)).$$

The inequality “ $\leq$ ” holds iff for each  $A' \in L^X$  such that  $A' = C(A')$  and  $S(A, A') \in K$  we have  $C(A)(x) \otimes S(A, A') \leq A'(x)$  which holds since  $C(A)(x) \otimes S(A, A') \leq C(A)(x) \otimes S(C(A), C(A')) \leq C(A')(x) = A'(x)$ . Conversely, putting  $A' = C(A)$  we get

$$S(A, C(A)) \rightarrow C(A)(x) = 1 \rightarrow C(A)(x) = C(A)(x),$$

i.e. “ $\geq$ ” also holds. ■

### **$L_K$ -closure systems as systems of “almost closed” $L$ -sets**

A natural idea is to consider the property “to be closed (w.r.t. a given fuzzy closure operator  $C$ )” a graded property. An  $L$ -set  $A$  can be considered to be “almost closed w.r.t.  $C$ ” iff “ $A$  almost equals  $C(A)$ ”. This poses a question of whether fuzzy closure systems can be defined as systems of “almost closed” fuzzy sets.

**DEFINITION 3.3.** An  $L$ -system  $\mathbf{S} \in L^{L^X}$  is called an  $L_K$ -closure  $L$ -system in  $X$  if for every  $A, B \in L^X$  we have

$$\mathbf{S}\left(\bigcap_{A_i \in L^X, S(A, A_i) \in K} (\mathbf{S}(A_i) \otimes S(A, A_i)) \rightarrow A_i\right) = 1, \quad (7)$$

$$\mathbf{S}(A) \otimes S(A, B) \otimes S(B, A) \leq \mathbf{S}(B) \quad (8)$$

$$\text{whenever } S(B, A) \in K. \quad (9)$$

*Remark 3.4.* (1) The  $L$ -set  $\bigcap_{A_i \in L^X, S(A, A_i) \in K} (\mathbf{S}(A_i) \otimes S(A, A_i)) \rightarrow A_i$  in  $X$  is defined by  $(\bigcap_{A_i \in L^X, S(A, A_i) \in K} (\mathbf{S}(A_i) \otimes S(A, A_i)) \rightarrow A_i)(x) = \bigwedge_{A_i \in L^X, S(A, A_i) \in K} (\mathbf{S}(A_i) \otimes S(A, A_i)) \rightarrow A_i(x)$ .

(2) An  $L_K$ -closure  $L$ -system is therefore an  $L$ -set of  $L$ -sets in  $X$ . We interpret  $\mathbf{S}(A)$  as the degree to which  $A \in L^X$  is closed. Condition (8) is naturally interpreted as the requirement that an  $L$ -set that is both a subset and a superset of to an “almost closed”  $L$ -set is itself “almost closed”.

We are going to investigate the relationship between  $\mathbf{L}_K$ -closure  $\mathbf{L}$ -systems,  $\mathbf{L}_K$ -closure operators, and  $\mathbf{L}_K$ -closure systems. To this end we define the following mappings.

For an  $\mathbf{L}_K$ -closure operator  $C$  in  $X$  and an  $\mathbf{L}_K$ -closure system  $\mathcal{S}$  in  $X$  we define  $\mathbf{L}$ -sets  $\mathbf{S}_C$  and  $\mathbf{S}_\mathcal{S}$  in  $L^X$  by

$$\mathbf{S}_C(A) = E(A, C(A)), \quad (10)$$

$$\mathbf{S}_\mathcal{S}(A) = E(A, C_\mathcal{S}(A)). \quad (11)$$

Clearly, we have  $\mathbf{S}_C(A) = S(C(A), A)$  and  $\mathbf{S}_\mathcal{S}(A) = S(C_\mathcal{S}(A), A)$ .

For an  $\mathbf{L}_K$ -closure  $\mathbf{L}$ -system  $\mathbf{S}$  in  $X$  we define a mapping  $C_\mathbf{S} : L^X \rightarrow L^X$  and a set  $\mathcal{S}_\mathbf{S} \subseteq L^X$  by

$$(C_\mathbf{S}(A))(x) = \bigwedge_{A_i \in L^X, S(A, A_i) \in K} (\mathbf{S}(A_i) \otimes S(A, A_i)) \rightarrow A_i(x), \quad (12)$$

$$\mathcal{S}_\mathbf{S} = \{A \in L^X \mid \mathbf{S}(A) = 1\}. \quad (13)$$

LEMMA 3.3. For an  $\mathbf{L}_K$ -closure operator  $C$  in  $X$  we have  $C_{\mathbf{S}_C} = C_{\mathcal{S}_C}$ .

*Proof.* Take any  $A \in L^X$ ,  $x \in X$ . We have to show  $(C_{\mathbf{S}_C}(A))(x) = (C_{\mathcal{S}_C}(A))(x)$ .

“ $\leq$ ”:

$$\begin{aligned} (C_{\mathbf{S}_C}(A))(x) &= \bigwedge_{A_i \in L^X, S(A, A_i) \in K} \mathbf{S}_C(A_i) \otimes S(A, A_i) \rightarrow A_i(x) \leq \\ &\leq \bigwedge_{A_i \in L^X, S(A, A_i) \in K, \mathbf{S}_C(A_i) = 1} \mathbf{S}_C(A_i) \otimes S(A, A_i) \rightarrow A_i(x) = \\ &= \bigwedge_{A_i \in L^X, S(A, A_i) \in K, A_i = C(A_i)} S(A, A_i) \rightarrow A_i(x) = (C_{\mathcal{S}_C}(A))(x). \end{aligned}$$

“ $\geq$ ”:

By definitions, the inequality holds iff

$$C_{\mathcal{S}_C}(A)(x) \leq \mathbf{S}_C(A_j) \otimes S(A, A_j) \rightarrow A_j(x)$$

for any  $j$  such that  $S(A, A_j) \in K$ . Since  $K$  is an  $\leq$ -filter in  $\mathbf{L}$ ,  $S(A, A_j) \in K$  and  $S(A, A_j) \leq S(A, C(A_j))$  imply  $S(A, C(A_j)) \in K$ . Therefore,  $C_{\mathcal{S}_C}(A)(x) =$

$$= \bigwedge_{A_i \in L^X, S(A, A_i) \in K, A_i = C(A_i)} S(A, A_i) \rightarrow A_i(x) \leq S(A, C(A_j)) \rightarrow C(A_j)(x).$$

It follows that it is sufficient to show

$$\mathbf{S}(A, C(A_j)) \rightarrow C(A_j)(x) \leq \mathbf{S}_C(A_j) \otimes \mathbf{S}(A, A_j) \rightarrow A_j(x).$$

The last inequality holds iff  $\mathbf{S}_C(A_j) \otimes \mathbf{S}(A, A_j) \otimes (\mathbf{S}(A, C(A_j)) \rightarrow C(A_j)(x)) \leq A_j(x)$  which is true. Indeed,

$$\begin{aligned} & \mathbf{S}_C(A_j) \otimes \mathbf{S}(A, A_j) \otimes (\mathbf{S}(A, C(A_j)) \rightarrow C(A_j)(x)) = \\ & = \mathbf{S}(C(A_j), A_j) \otimes \mathbf{S}(A, A_j) \otimes (\mathbf{S}(A, C(A_j)) \rightarrow C(A_j)(x)) \leq \\ & \leq \mathbf{S}(C(A_j), A_j) \otimes \mathbf{S}(A, C(A_j)) \otimes (\mathbf{S}(A, C(A_j)) \rightarrow C(A_j)(x)) \leq \\ & \leq \mathbf{S}(C(A_j), A_j) \otimes C(A_j)(x) \leq A_j(x). \end{aligned}$$

■

LEMMA 3.4. *For any  $\mathbf{L}_K$ -closure operator  $C$  in  $X$ ,  $\mathbf{S}_C$  is an  $\mathbf{L}_K$ -closure  $\mathbf{L}$ -system in  $X$ .*

*Proof.* We verify (7) and (8).

(7): We have to show that for any  $A \in L^X$  we have

$$\mathbf{S}_C\left(\bigcap_{A_i \in L^X, \mathbf{S}(A, A_i) \in K} \mathbf{S}_C(A_i) \rightarrow A_i(x)\right) = 1,$$

i.e.  $\mathbf{S}_C(\mathbf{C}_{\mathbf{S}_C}(A)) = 1$ , i.e.  $C(\mathbf{C}_{\mathbf{S}_C}(A)) = \mathbf{C}_{\mathbf{S}_C}(A)$ . The last equality, however, follows from idempotency of  $C$  by observing that  $\mathbf{C}_{\mathbf{S}_C} = C$  (Lemma 3.3).

(8): We have to show  $\mathbf{S}(A) \otimes \mathbf{S}(A, B) \otimes \mathbf{S}(B, A) \leq \mathbf{S}(B)$ , i.e.

$$\mathbf{S}(C(A), A) \otimes \mathbf{S}(A, B) \otimes \mathbf{S}(B, A) \leq \mathbf{S}(C(B), B)$$

which holds iff for each  $x \in X$  we have  $C(B)(x) \otimes \mathbf{S}(B, A) \otimes \mathbf{S}(C(A), A) \otimes \mathbf{S}(A, B) \leq B(x)$ . The last inequality is true:

$$\begin{aligned} & C(B)(x) \otimes \mathbf{S}(B, A) \otimes \mathbf{S}(C(A), A) \otimes \mathbf{S}(A, B) \leq \\ & \leq C(B)(x) \otimes \mathbf{S}(C(B), C(A)) \otimes \mathbf{S}(C(A), A) \otimes \mathbf{S}(A, B) \leq \\ & \leq C(B)(x) \otimes \mathbf{S}(C(B), B) \leq B(x). \end{aligned}$$

■

LEMMA 3.5. *Let  $K$  be a filter in  $\mathbf{L}$ . For any  $\mathbf{L}_K$ -closure  $\mathbf{L}$ -system  $\mathbf{S}$  in  $X$ ,  $\mathbf{C}_{\mathbf{S}}$  is an  $\mathbf{L}_K$ -closure operator in  $X$ .*

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*Proof.* We verify (2)–(4).

(2):  $A \subseteq C_{\mathbf{S}}(A)$  holds iff  $A(x) \otimes S(A, A_i) \otimes \mathbf{S}(A_i) \leq A_i(x)$  for any  $i$  such that  $S(A, A_i) \in K$  which is evidently true since

$$A(x) \otimes S(A, A_i) \otimes \mathbf{S}(A_i) \leq A_i(x) \otimes \mathbf{S}(A_i) \leq A_i(x).$$

(3): Let  $S(A, B) \in K$ .  $S(A, B) \leq S(C_{\mathbf{S}}(A), C_{\mathbf{S}}(B))$  is true iff for each  $x \in X$  we have  $S(A, B) \otimes C_{\mathbf{S}}(A)(x) \leq C_{\mathbf{S}}(B)(x)$  iff for any  $A_j \in L^X$  such that  $S(B, A_j) \in K$  we have

$$\begin{aligned} & \mathbf{S}(A_j) \otimes S(B, A_j) \otimes S(A, B) \otimes \\ & \otimes \left( \bigwedge_{A_i \in L^X, S(A, A_i) \in K} \mathbf{S}(A_i) \otimes S(A, A_i) \rightarrow A_i(x) \right) \leq A_j(x). \end{aligned}$$

The last inequality is true: since  $S(A, B) \otimes S(B, A_j) \leq S(A, A_j)$ ,  $S(A, B) \in K$  and  $S(B, A_j) \in K$  yield  $S(A, A_j) \in K$ . Therefore,

$$\begin{aligned} & \mathbf{S}(A_j) \otimes S(B, A_j) \otimes S(A, B) \otimes \\ & \otimes \left( \bigwedge_{A_i \in L^X, S(A, A_i) \in K} \mathbf{S}(A_i) \otimes S(A, A_i) \rightarrow A_i(x) \right) \leq \\ & \leq \mathbf{S}(A_j) \otimes S(A, A_j) \otimes (\mathbf{S}(A_j) \otimes S(A, A_j) \rightarrow A_j(x)) \leq A_j(x). \end{aligned}$$

(4):

$$\begin{aligned} & C_{\mathbf{S}}(C_{\mathbf{S}}(A))(x) = \\ & = \bigwedge_{A_i \in L^X, S(C_{\mathbf{S}}(A), A_i) \in K} \mathbf{S}(A_i) \otimes S(C_{\mathbf{S}}(A), A_i) \rightarrow A_i(x) \leq \\ & \leq \mathbf{S}(C_{\mathbf{S}}(A)) \otimes S(C_{\mathbf{S}}(A), C_{\mathbf{S}}(A)) \rightarrow C_{\mathbf{S}}(A)(x) = 1 \rightarrow C_{\mathbf{S}}(A)(x) = \\ & = C_{\mathbf{S}}(A)(x). \end{aligned}$$

■

The relationship between  $\mathbf{L}_K$ -closure operators,  $\mathbf{L}_K$ -closure systems, and  $\mathbf{L}_K$ -closure  $\mathbf{L}$ -systems is the subject of the following theorems.

**THEOREM 3.5.** *Let  $C$  be an  $\mathbf{L}_K$ -closure operator in  $X$ ,  $\mathbf{S}$  be an  $\mathbf{L}_K$ -closure  $\mathbf{L}$ -system,  $K$  be a filter in  $\mathbf{L}$ . Then  $\mathbf{S}_C$  is an  $\mathbf{L}_K$ -closure  $\mathbf{L}$ -system in  $X$ ,  $C_{\mathbf{S}}$  is an  $\mathbf{L}_K$ -closure operator in  $X$ , and  $C = C_{\mathbf{S}_C}$  and  $\mathbf{S} = \mathbf{S}_{C_{\mathbf{S}}}$ , i.e. the mappings  $C \mapsto \mathbf{S}_C$  and  $\mathbf{S} \mapsto C_{\mathbf{S}}$  are mutually inverse.*

*Proof.* By Lemma 3.4 and Lemma 3.5, it remains to verify  $C = C_{S_C}$  and  $S = S_{C_S}$ . By Lemma 3.3 and by Theorem 3.4,  $C_{S_C} = C_{S_C} = C$ . Since  $S_{C_S}(A) = S(C_S(A), A)$ , it remains to prove  $S(A) = S(C_S(A), A)$ :

On the one hand,  $S(A) \leq S(C_S(A), A)$  iff for each  $x \in X$  we have  $S(A) \otimes C_S(A)(x) \leq A(x)$ , i.e.

$$S(A) \otimes \bigwedge_{A_i \in L^X, S(A, A_i) \in K} (S(A_i) \otimes S(A, A_i) \rightarrow A_i(x)) \leq A(x)$$

which is true since

$$\begin{aligned} S(A) \otimes \bigwedge_{A_i \in L^X, S(A, A_i) \in K} (S(A_i) \otimes S(A, A_i) \rightarrow A_i(x)) &\leq \\ \leq S(A) \otimes (S(A) \otimes S(A, A) \rightarrow A(x)) &\leq A(x). \end{aligned}$$

On the other hand,

$$S(C_S(A), A) = S(C_S(A)) \otimes S(C_S(A), A) \otimes S(A, C_S(A)) \leq S(A),$$

by (8). ■

**THEOREM 3.6.** *Let  $S$  be an  $L_K$ -closure system in  $X$ ,  $S$  be an  $L_K$ -closure  $L$ -system,  $K$  be a filter in  $L$ . Then  $S_S$  is an  $L_K$ -closure  $L$ -system in  $X$ ,  $S_S$  is an  $L_K$ -closure system in  $X$ , and  $S = S_{S_S}$  and  $S = S_{S_S}$ , i.e. the mappings  $S \mapsto S_S$  and  $S \mapsto S_S$  are mutually inverse.*

*Proof.* By definition,  $S_S = S_{C_S}$ , therefore, by Lemma 3.4,  $S_S$  is an  $L_K$ -closure  $L$ -system. To see that  $S_S$  is an  $L_K$ -closure system it is, by Theorem 3.4, sufficient to show  $S_S = S_{C_S}$ , i.e.

$$\{A \in L^X \mid S(A) = 1\} = \{A \in L^X \mid A = C_S(A)\}.$$

On the one hand,  $S(A) = 1$  implies

$$\begin{aligned} C_S(A)(x) &= \bigwedge_{A_i \in L^X, S(A, A_i)} S(A_i) \otimes S(A, A_i) \rightarrow A_i(x) \leq \\ &\leq S(A) \otimes S(A, A) \rightarrow A(x) = A(x), \end{aligned}$$

i.e.  $A = C_S(A)$ . On the other hand,  $A = C_S(A)$  implies (using (7))  $1 = S(C_S(A)) = S(A)$ . Therefore,  $S_S = S_{C_S}$ .

We show  $S = S_{S_S}$ : We have  $A \in S$  iff  $A = C_S(A)$  iff  $S_{C_S}(A) = 1$  iff  $S_S(A) = 1$  iff  $A \in S_{S_S}$ . It remains to show  $S(A) = S_{S_S}(A)$ : We have  $C_S =$

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$C_{S_{C_S}}$  and (by the above observation)  $S_{C_S} = S_S$ . Therefore,  $C_S = C_{S_S}$ . Using  $\mathbf{S}(A) = S(C_S(A), A)$  (see the end of the proof of Theorem 3.5), we conclude  $\mathbf{S}(A) = S(C_S(A), A) = S(C_{S_S}(A), A) = \mathbf{S}_{S_S}(A)$  completing the proof. ■

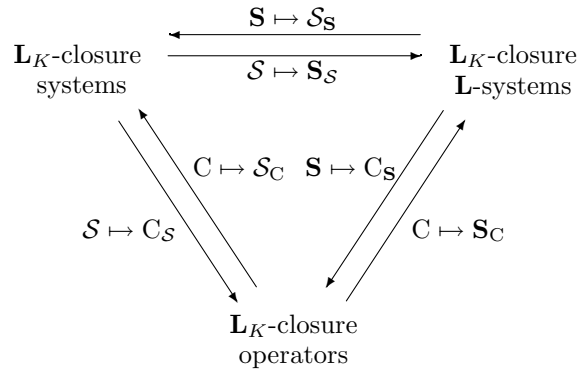


FIG. 1. Commuting diagram of Corollary 3.2

COROLLARY 3.2. *Under the above introduced notation, the diagram in Fig. 1 commutes.*

*Proof.* Each oriented path in the diagram in Fig. 1 defines a mapping (that one composed of the mappings represented by the arrows). The assertion says that any two mappings corresponding to oriented paths with common starting and final node are equal. Using Theorem 3.4, Theorem 3.5, and Theorem 3.6, a moment reflection shows that it is sufficient to show  $S_S = S_{C_S}$  and  $S_S = S_{C_S}$ . However, both of these identities were established in the proof of Theorem 3.6. ■

### $L_K$ -Galois connections

We are going to investigate the relationship of fuzzy closure operators to fuzzy Galois connections.

DEFINITION 3.4. An  $L_K$ -Galois connection (fuzzy Galois connection) between the sets  $X$  and  $Y$  is a pair  $\langle \uparrow, \downarrow \rangle$  of mappings  $\uparrow : L^X \rightarrow L^Y$ ,  $\downarrow : L^Y \rightarrow L^X$ , satisfying

$$S(A_1, A_2) \leq S(A_2^\uparrow, A_1^\uparrow) \quad \text{whenever } S(A_1, A_2) \in K \quad (14)$$

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$$S(B_1, B_2) \leq S(B_2^\downarrow, B_1^\downarrow) \quad \text{whenever } S(B_1, B_2) \in K \quad (15)$$

$$A \subseteq (A^\uparrow)^\downarrow \quad (16)$$

$$B \subseteq (B^\downarrow)^\uparrow. \quad (17)$$

for every  $A, A_1, A_2 \in L^X$ ,  $B, B_1, B_2 \in L^Y$ .

If  $K = L$  then we again omit the subscript  $K$ . Note also that an  $\mathbf{L}_K$ -Galois connection between  $X$  and  $Y$  forms a Galois connection between the complete lattices  $\langle L^X, \subseteq \rangle$  and  $\langle L^Y, \subseteq \rangle$  [5, 11].

*Remark 3. 5.* Note that Galois connections between sets [5, 11] are just  $\mathbf{L}$ -Galois connections for  $\mathbf{L} = \mathbf{2}$ .

For the following simple characterization see [1].

**THEOREM 3.7.** *A pair  $\langle \uparrow, \downarrow \rangle$  forms an  $\mathbf{L}_K$ -Galois connection between  $X$  and  $Y$  iff  $S(A, B^\downarrow) \in K$  or  $S(B, A^\uparrow) \in K$  implies*

$$S(A, B^\downarrow) = S(B, A^\uparrow) \quad (18)$$

for all  $A \in L^X$ ,  $B \in L^Y$ .

Call two systems  $\mathcal{S}_1 \subseteq L^X$  and  $\mathcal{S}_2 \subseteq L^Y$  of  $\mathbf{L}$ -sets in  $X$  and  $\mathbf{L}$ -sets in  $Y$ , respectively,  $S_K$ -dually isomorphic iff there is a bijective mapping  $\varphi : \mathcal{S}_1 \rightarrow \mathcal{S}_2$  such that for  $A_1, A_2 \in \mathcal{S}_1$ ,  $B_1 = \varphi(A_1), B_2 = \varphi(A_2) \in \mathcal{S}_2$  it holds that  $S(A_1, A_2) = S(B_2, B_1)$  whenever  $S(A_1, A_2) \in K$  or  $S(B_2, B_1) \in K$ .

**THEOREM 3.8.** *Let  $\langle \uparrow, \downarrow \rangle$  be an  $\mathbf{L}_K$ -Galois connection between  $X$  and  $Y$ , let  $C^X$  and  $C^Y$  be  $\mathbf{L}_K$ -closure operators on  $X$  and  $Y$ , respectively, such that  $S_{C^X}$  and  $S_{C^Y}$  are  $S_K$ -dually isomorphic with  $\varphi$  being the isomorphism. Put  $C_{\langle \uparrow, \downarrow \rangle}^X = \uparrow\downarrow$  and  $C_{\langle \uparrow, \downarrow \rangle}^Y = \downarrow\uparrow$  (the composite mappings), and let  $(A)_{\langle C^X, C^Y \rangle}^\uparrow = \varphi(C(A))$  and  $(B)_{\langle C^X, C^Y \rangle}^\downarrow = \varphi^{-1}(C(B))$  for  $A \in L^X$ ,  $B \in L^Y$ . Then the following is true.*

(1)  $C_{\langle \uparrow, \downarrow \rangle}^X$  and  $C_{\langle \uparrow, \downarrow \rangle}^Y$  are  $\mathbf{L}_K$ -closure operators on  $X$  and  $Y$ , respectively, and  $S_{C_{\langle \uparrow, \downarrow \rangle}^X}$  and  $S_{C_{\langle \uparrow, \downarrow \rangle}^Y}$  are  $S_K$ -dually isomorphic.

(2)  $\langle \uparrow_{\langle C^X, C^Y \rangle}, \downarrow_{\langle C^X, C^Y \rangle} \rangle$  is an  $\mathbf{L}_K$ -Galois connection.

(3) The correspondences defined by  $\langle \uparrow, \downarrow \rangle \mapsto \langle C_{\langle \uparrow, \downarrow \rangle}^X, C_{\langle \uparrow, \downarrow \rangle}^Y \rangle$  and  $\langle C^X, C^Y \rangle \mapsto \langle \uparrow_{\langle C^X, C^Y \rangle}, \downarrow_{\langle C^X, C^Y \rangle} \rangle$  are mutually inverse mappings.



*Proof.* (1): (16) implies (2). If  $S(A_1, A_2) \in K$  then, by (14), by the  $\leq$ -filter property of  $K$ , and by (15), we have

$$S(A_1, A_2) \leq S(A_2^\uparrow, A_1^\uparrow) \leq S(A_1^{\uparrow\downarrow}, A_2^{\uparrow\downarrow}),$$

i.e. (3) holds. The fact that  $\langle \uparrow, \downarrow \rangle$  is a Galois connection between the lattices  $L^X$  and  $L^Y$  immediately gives (4) [5], i.e.  $C_{\langle \uparrow, \downarrow \rangle}^X$  is an  $\mathbf{L}_K$ -closure operator on  $X$ . The proof for  $C_{\langle \downarrow, \uparrow \rangle}^Y$  is completely analogous. The rest easily follows by Theorem 3.7 observing that  $A = A^{\uparrow\downarrow}$  and  $B = B^{\downarrow\uparrow}$  for  $A \in \mathcal{S}_{C_{\langle \uparrow, \downarrow \rangle}^X}$  and  $B \in \mathcal{S}_{C_{\langle \downarrow, \uparrow \rangle}^Y}$ .

(2): For simplicity, we write only  $\uparrow$  instead of  $\uparrow_{\langle C^X, C^Y \rangle}$ , the same for  $\downarrow$ . First, since

$$C^X(A) = \varphi^{-1}(\varphi(C^X(A))) = \varphi^{-1}(C^Y(\varphi(C^X(A)))) = A^{\uparrow\downarrow},$$

we have  $C^X = C_{\langle \uparrow, \downarrow \rangle}^X$  whence (14) follows. If  $S(A_1, A_2) \in K$  then by (3) and the assumption,

$$S(A_1, A_2) \leq S(C^X(A_1), C^X(A_2)) = S(\varphi(C^X(A_2)), \varphi(C^X(A_1))) = S(A_2^\uparrow, A_1^\uparrow),$$

i.e. (15) holds. The rest of the statement can be proved analogously.

Part (3) now follows easily from the proof of (2) and the fact that  $A^\uparrow = A^{\uparrow\downarrow\uparrow}$  and  $B^\downarrow = B^{\downarrow\uparrow\downarrow}$  for  $A \in L^X, B \in L^Y$ . ■

For  $K = L$  the foregoing theorem can be strengthened.

**THEOREM 3.9.** *Let  $C$  be an  $\mathbf{L}$ -closure operator, and  $Y = \{C(A) \mid A \in L^X\}$ . Then the pair of mappings  $\uparrow^C : L^X \rightarrow L^Y, \downarrow^C : L^Y \rightarrow L^X$  defined for  $A \in L^X, B \in L^Y$  and  $x \in X, A' \in Y$  by*

$$\begin{aligned} A^{\uparrow^C}(A') &= S(A, A') \\ B^{\downarrow^C}(x) &= \bigwedge_{A \in Y, B(A) \in K} B(A) \rightarrow (A)(x) \end{aligned}$$

*forms an  $\mathbf{L}$ -Galois connection such that  $C = \uparrow^C \downarrow^C$ .*

*Proof.* For brevity we write  $\uparrow$  and  $\downarrow$  instead of  $\uparrow^C$  and  $\downarrow^C$ , respectively. We first verify (14)–(17).

(14):  $S(A_1, A_2) \leq S(A_1^\uparrow, A_2^\uparrow)$  holds iff for each  $A' \in Y$  it holds  $S(A_1, A_2) \leq S(A_2, A') \rightarrow S(A_1, A')$  which holds iff  $S(A_1, A_2) \otimes S(A_2, A') \leq S(A_1, A')$  iff for each  $x \in X$  it holds  $A_1(x) \otimes S(A_1, A_2) \otimes S(A_2, A') \leq A'(x)$  which is

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true since

$$\begin{aligned} & A_1(x) \otimes S(A_1, A_2) \otimes S(A_2, A') \leq \\ & \leq A_1(x) \otimes (A_1(x) \rightarrow A_2(x)) \otimes (A_2(x) \rightarrow A'(x)) \leq A'(x). \end{aligned}$$

(15): Let  $B_1, B_2 \in L^Y$ . We have to prove  $S(B_1, B_2) \leq S(B_2^\downarrow, B_1^\downarrow)$  which holds iff for each  $x \in X$  it holds

$$S(B_1, B_2) \otimes B_2^\downarrow(x) \leq B_1^\downarrow(x) = \bigwedge_{A \in Y} (B_1(A) \rightarrow A(x))$$

iff for each  $A \in Y$  it holds

$$B_1(A) \otimes S(B_1, B_2) \otimes B_2^\downarrow(x) \leq A(x)$$

which is valid since

$$\begin{aligned} & B_1(A) \otimes S(B_1, B_2) \otimes B_2^\downarrow(x) \leq \\ & \leq B_1(A) \otimes (B_1(A) \rightarrow B_2(A)) \otimes (B_2(A) \rightarrow A(x)) \leq A(x). \end{aligned}$$

To show (16), it suffices to show that  $C = \uparrow^\downarrow$ . Let thus  $A \in L^X$ ,  $x \in X$ . We show  $C(A)(x) = A^{\uparrow\downarrow}(x)$  by proving both of the inequalities.

“ $\leq$ ”:  $C(A)(x) \leq A^{\uparrow\downarrow}(x) = \bigwedge_{A' \in Y} (A^\uparrow(A') \rightarrow A'(x))$  holds iff for each  $A' \in Y$  it holds  $A^\uparrow(A') \otimes C(A)(x) \leq A'(x)$  which holds iff  $A^\uparrow(A') \leq C(A)(x) \rightarrow A'(x)$ , i.e.  $S(A, A') \leq C(A)(x) \rightarrow A'(x)$  which holds since

$$\begin{aligned} & S(A, A') \leq S(C(A), C(A')) \leq \\ & \leq C(A)(x) \rightarrow C(A')(x) = C(A)(x) \rightarrow A'(x) \end{aligned}$$

as  $C(A') = A'$ .

“ $\geq$ ”:  $A^{\uparrow\downarrow}(x) \leq C(A)(x)$ , i.e.  $\bigwedge_{A' \in Y} (S(A, A') \rightarrow A'(x)) \leq C(A)(x)$  holds since for  $A' = C(A)$  we have

$$S(A, C(A)) \rightarrow C(A)(x) = 1 \rightarrow C(A)(x) = C(A)(x).$$

(17): We have  $B(A) \leq B^{\downarrow\uparrow}(A)$  iff

$$B(A) \leq S(B^\downarrow, A) = \bigwedge_{x \in X} (B^\downarrow(x) \rightarrow A(x))$$

which holds iff for each  $x \in X$  we have  $B(A) \otimes B^\downarrow(x) \leq A(x)$ , i.e.

$$B(A) \otimes \bigwedge_{A' \in Y} (B(A') \rightarrow A'(x)) \leq A(x)$$

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which holds (putting  $A' = A$  the inequality is evident). The proof is complete. ■

Further results and some natural examples of fuzzy closure operators can be found in [2] which is meant to be a follow-up to this paper.

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