Fuzzy closure operators II: induced relations, representation, and examples

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Abstract. Closure operators (and related structures) are investigated from the point of view of fuzzy set theory. The paper is a follow up to [7] where fundamental notions and result have been established. The present approach generalizes the existing approaches in two ways: First, complete residuated lattices are used as the structures of truth values (leaving the unite interval [0,1] with minimum and other t-norms particualr cases); second, the monotony condition is formulated so that it can reflect also partial subsethood (not only full subsethood as in other approaches).

In this paper, we study relations induced by fuzzy closure operators (fuzzy quasiorders and similarities); factorization of closure systems by similarities and by so-called decrease of logical precision; representation of fuzzy closure operators by (crisp) closure operators; relation to consequence relations; and natural examples illustrating the notions and results.

1 Introduction

This is a follow up to my paper [7]. In [7], closure operators and related structures have been considered from the point of view of fuzzy approach (graded truth approach; with complete residuated lattices taken for the structures of truth values). The aim of this paper is to present further results on fuzzy closure operators.

The organization and the content of the paper are as follows: Section 2 recalls the notions and main results of [7]. In Section 3, we study some induced (fuzzy) relations: fuzzy quasiorder and equivalence (similarity). We show a way to factorize the complete lattice of closed (w.r.t. to a given fuzzy closure operator) fuzzy sets by an a-cut of a naturally defined similarity relation, parameter a having the role of controlling the coarsity of the factorization. Another way to factorize the lattice of closed fuzzy sets is by a so-called decrease of logical precision. The factorization processes have natural applications if the structure of closed sets has some natural interpretation and one needs to simplify the structure (as an example, we demonstrate the results on factorization of so-called fuzzy concept lattices). In Section 4, we present a natural representation of fuzzy closure operators by (classical) closure operators. Section 5 presents some examples of fuzzy closure operators. In Section 6, fuzzy closure operators and consequence relations are briefly discussed.

2 Fuzzy closure operators

Closure operators (and closure systems) play a significant role in both pure and applied mathematics. In the framework of fuzzy set theory, several particular examples of closure operators and systems have been considered (e.g. so-called fuzzy subalgebras, fuzzy congruences, fuzzy topology etc.). Recently, fuzzy closure operators and fuzzy closure systems themselves have been studied, see e.g. [8, 9, 15, 16]. As a matter of fact, a fuzzy set A is usually defined as a mapping from a universe set X into the real interval $[0,1]$ in the above mentioned works. Therefore, the structure of truth values of the "logic behind" is fixed to $[0,1]$ equipped with minimum being the operation corresponding to logical conjunction.

A general approach to the study of fuzzy closure operators has been outlined in [7]. Compared to previous approaches, there are basically two points of departure: First, the structure of truth values is assumed to form a complete residuated lattice. Second, the monotonicity condition is defined to mean "if A is almost a subset of B then the closure of A is almost a subset of the closure of B".

We now recall basic concepts and results (for proofs and further results we refer to [7]).

Definition 1 *A* complete residuated lattice *is an algebra* $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ *such that*

(1) $\langle L, \wedge, \vee, 0, 1\rangle$ *is a complete lattice with the least element* 0 *and the greatest element* 1*;*

(2) $\langle L, \otimes, 1\rangle$ *is a commutative monoid, i.e.* $\otimes$ *is commutative, associative, and* $x \otimes 1 = x$ *holds holds for each* $x \in L$*;*

(3) $\otimes, \rightarrow$ *form an adjoint pair, i.e.*

$$x \otimes y \leq z \text{ iff } x \leq y \rightarrow z \tag{1}$$

holds for all $x, y, z \in L$.

Residuated lattices play the role of structures of truth values in fuzzy logic. Introduced originally in the study of ideal systems of rings [24], residuated lattices have been introduced into the context of fuzzy logic by Goguen [17]. For logical calculi with truth values in residuated lattices (and special types of residuated lattices), basic properties of residuated lattices, and further references we refer to [18, 19, 20].

We only recall that the most studied and applied residuated lattices are those defined on the real interval $[0,1]$ (residuated lattices on $[0,1]$ uniquely correspond to left-continuous t-norms). Three most important structures pairs of adjoint operations are the following: the Łukasiewicz one ($a \otimes b = \max(a+b-1, 0)$, $a \rightarrow b = \min(1-a+b, 1)$), Gödel one ($a \otimes b = \min(a,b)$, $a \rightarrow b = 1$ if $a \leq b$ and $= b$ else), and product one ($a \otimes b = a \cdot b$, $a \rightarrow b = 1$ if $a \leq b$ and $= b/a$

else). Another important set of truth values is the set $\{a_0 = 0, a_1, \ldots, a_n = 1\}$ ($a_0 < \cdots < a_n$) with $\otimes$ given by $a_k \otimes a_l = a_{\max(k+l-n,0)}$ and the corresponding $\rightarrow$ given by $a_k \rightarrow a_l = a_{\min(n-k+l,n)}$. A special case of the latter algebras is the Boolean algebra $\mathbf{2}$ of classical logic with the support $2 = \{0, 1\}$.

A nonempty subset $K \subseteq L$ is called an *$\leq$-filter* if for every $a, b \in L$ such that $a \leq b$ it holds that $b \in K$ whenever $a \in K$. An $\leq$-filter K is called a *filter* if $a, b \in K$ implies $a \otimes b \in K$. Unless otherwise stated, in what follows we denote by $\mathbf{L}$ a complete residuated lattice and by K an $\leq$-filter in $\mathbf{L}$ (both $\mathbf{L}$ and K possibly with indices).

An $\mathbf{L}$-*set* (fuzzy set) [28, 17] A in a universe set X is any map $A : X \rightarrow L$. By L^X we denote the set of all $\mathbf{L}$-sets in X. The concept of an $\mathbf{L}$-relation is defined obviously. Operations on L extend pointwise to L^X, e.g. $(A \vee B)(x) = A(x) \vee B(x)$ for $A, B \in L^X$. Following common usage, we write $A \cup B$ instead of $A \vee B$, etc. Given $A, B \in L^X$, the *subsethood degree* [17] $\mathrm{S}(A, B)$ of A in B is defined by $\mathrm{S}(A, B) = \bigwedge_{x \in X} A(x) \rightarrow B(x)$. We write $A \subseteq B$ if $\mathrm{S}(A, B) = 1$. Analogously, the *equality degree* $E(A, B)$ of A and B is defined by $E(A, B) = \bigwedge_{x \in X}(A(x) \leftrightarrow B(x))$. It is immediate that $E(A, B) = \mathrm{S}(A, B) \wedge \mathrm{S}(B, A)$. By $\{{}^{a_1}/x_1, \cdots, {}^{a_n}/x_n\}$ we denote an $\mathbf{L}$-set A for which $A(x) = a_i$ if $x = x_i$ ($i = 1, \ldots, n$) and $A(x) = 0$ otherwise.

Recall that a (classical) closure operator on a set X is a mapping $\mathrm{C} : 2^X \rightarrow 2^X$ satisfying the following conditions: $A \subseteq \mathrm{C}(A)$, if $A_1 \subseteq A_2$ then $\mathrm{C}(A_1) \subseteq \mathrm{C}(A_2)$, and $\mathrm{C}(A) = \mathrm{C}(\mathrm{C}(A))$, for any $A, A_1, A_2 \in 2^X$. More generally, if $\subseteq$ denotes a partial order, we get the notion of closure operator in an ordered set [10].

Definition 2 *An* $\mathbf{L}_K$-closure operator *(*fuzzy closure operator*) on the set X is a mapping $\mathrm{C} : L^X \rightarrow L^X$ satisfying*

$$\begin{aligned} A &\subseteq \mathrm{C}(A) && (2)\\ \mathrm{S}(A_1, A_2) &\leq \mathrm{S}(\mathrm{C}(A_1), \mathrm{C}(A_2)) \quad \textit{whenever } \mathrm{S}(A_1, A_2) \in K && (3)\\ \mathrm{C}(A) &= \mathrm{C}(\mathrm{C}(A)) && (4) \end{aligned}$$

for every $A, A_1, A_2 \in L^X$.

If $K = L$, we omit the subscript K and call C an $\mathbf{L}$-closure operator. The set K plays the role of the set of designated truth values. Condition (3) says that the closure preserves also partial subsethood whenever the subsethood degree is designated. Note that for $L = \{0, 1\}$, $\mathbf{L}_K$-closure operators coincide with (classical) closure operators. Note also that for $L = [0, 1]$, $\mathbf{L}_{\{1\}}$-closure operators are precisely fuzzy closure operators studied by Gerla [9, 15, 16].

Definition 3 *A system $\mathcal{S} = \{A_i \in L^X \mid i \in I\}$ is called* closed under S_K-intersections *iff for each $A \in L^X$ it holds that*

$$\Big(\bigcap_{i \in I, \mathrm{S}(A, A_i) \in K} \mathrm{S}(A, A_i) \rightarrow A_i\Big) \in \mathcal{S}$$

where

$$(\bigcap_{i\in I, \mathrm{S}(A,A_i)\in K} \mathrm{S}(A,A_i) \to A_i)(x) = \bigwedge_{i\in I, \mathrm{S}(A,A_i)\in K} (\mathrm{S}(A,A_i) \to A_i(x))$$

*for each $x \in X$. A system closed under S_K-intersections will be called an $\mathbf{L}_K$-*closure system.

For $K = L$ the subscript will again be omitted. **2**-closure systems coincide with closure systems, i.e. systems of sets closed under intersections. In general, being closed under arbitrary intersections is a weaker condition then being closed under S_K-intersections. Closedness under S_K-intersections is, however, equivalent to closedness under intersections of "K-shifted" **L**-sets. Let for $a \in L$, $A \in L^X$, denote by $a \to A$ the **L**-set defined by $(a \to A)(x) = a \to A(x)$.

Theorem 4 **([7])** *$\mathcal{S}$ is an $\mathbf{L}_K$-closure system iff for any $a_i \in L$, $i \in I$, it holds $\bigcap_{a_i\in K}(a_i \to A_i) \in \mathcal{S}$. Therefore, a system $\mathcal{S}$ which is closed under arbitrary intersections is an $\mathbf{L}_K$-closure system iff for each $a \in K$ and $A \in \mathcal{S}$ it holds $a \to A \in \mathcal{S}$.*

The following theorem shows another way to obtain the closure in an $\mathbf{L}_K$-closure system.

Theorem 5 **([7])** *Let $\mathcal{S} = \{A_i \in L^X \mid i \in I\}$ be an $\mathbf{L}_K$-closure system. Then for each $A \in L^X$ it holds*

$$\bigcap_{i\in I, \mathrm{S}(A,A_i)\in K} \mathrm{S}(A,A_i) \to A_i = \bigcap_{i\in I, A\subseteq A_i} A_i.$$

A natural idea is to consider the property "to be closed (w.r.t. a given fuzzy closure operator C)" a graded property. An **L**-set A can be considered to be "almost closed w.r.t. C" iff "A almost equals $\mathrm{C}(A)$". This poses a question of whether fuzzy closure systems can be defined as systems of "almost closed" fuzzy sets.

Definition 6 *An **L**-system $\mathbf{S} \in L^{L^X}$ is called an $\mathbf{L}_K$-*closure **L**-system in X *if for every $A, B \in L^X$ we have*

$$\mathbf{S}(\bigcap_{A_i\in L^X, \mathrm{S}(A,A_i)\in K} (\mathbf{S}(A_i) \otimes \mathrm{S}(A,A_i)) \to A_i) = 1, \tag{5}$$

$$\mathbf{S}(A) \otimes \mathrm{S}(A,B) \otimes \mathrm{S}(B,A) \leq \mathbf{S}(B) \tag{6}$$

$$\textit{whenever } \mathrm{S}(B,A) \in K. \tag{7}$$

Remark (1) Note that the **L**-set

$$\bigcap_{A_i\in L^X, \mathrm{S}(A,A_i)\in K} (\mathbf{S}(A_i) \otimes \mathrm{S}(A,A_i)) \to A_i$$

$$
\begin{tikzcd}
\text{$\mathbf{L}_K$-closure systems} \arrow[rr, shift right, "\mathcal{S} \mapsto \mathbf{S}_{\mathcal{S}}"'] \arrow[dr, shift right, "\mathcal{S} \mapsto \mathrm{C}_{\mathcal{S}}"'] & & \text{$\mathbf{L}_K$-closure $\mathbf{L}$-systems} \arrow[ll, shift right, "\mathbf{S} \mapsto \mathcal{S}_{\mathbf{S}}"'] \arrow[dl, shift right, "\mathbf{S} \mapsto \mathrm{C}_{\mathbf{S}}"'] \\
& \text{$\mathbf{L}_K$-closure operators} \arrow[ul, shift right, "\mathrm{C} \mapsto \mathcal{S}_{\mathrm{C}}"'] \arrow[ur, shift right, "\mathrm{C} \mapsto \mathbf{S}_{\mathrm{C}}"'] &
\end{tikzcd}
$$

Figure 1: Commuting diagram of Theorem 7

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in X is defined by $(\bigcap_{A_i \in L^X, \mathrm{S}(A,A_i) \in K}(\mathbf{S}(A_i) \otimes \mathrm{S}(A, A_i)) \to A_i)(x) = \bigwedge_{A_i \in L^X, \mathrm{S}(A,A_i) \in K}(\mathbf{S}(A_i) \otimes \mathrm{S}(A, A_i)) \to A_i(x)$.

(2) An $\mathbf{L}_K$-closure $\mathbf{L}$-system is therefore an $\mathbf{L}$-set of $\mathbf{L}$-sets in X. We interpret $\mathbf{S}(A)$ as the degree to which $A \in L^X$ is closed. Condition (6) is naturally interpreted as the requirement that an $\mathbf{L}$-set that is both a subset and a superset of to an "almost closed" $\mathbf{L}$-set is itself "almost closed".

Let C be an $\mathbf{L}_K$-closure operator in X, $\mathcal{S}$ be an $\mathbf{L}_K$-closure system in X, and $\mathbf{S}$ be an $\mathbf{L}_K$-closure $\mathbf{L}$-system in X. Define operators $\mathrm{C}_{\mathcal{S}} : L^X \to L^X$ and $\mathrm{C}_{\mathbf{S}} : L^X \to L^X$, systems of $\mathbf{L}$-sets $\mathcal{S}_{\mathrm{C}} \subseteq L^X$ and $\mathcal{S}_{\mathbf{S}} \subseteq L^X$, and $\mathbf{L}$-systems of $\mathbf{L}$-sets $\mathbf{S}_{\mathrm{C}} \in L^{L^X}$ and $\mathbf{S}_{\mathcal{S}} \in L^{L^X}$ by

$$\mathrm{C}_{\mathcal{S}}(A)(x) = \bigwedge_{i \in I, \mathrm{S}(A,A_i) \in K} (\mathrm{S}(A, A_i) \to A_i(x)) \tag{8}$$

$$(\mathrm{C}_{\mathbf{S}}(A))(x) = \bigwedge_{A_i \in L^X, \mathrm{S}(A,A_i) \in K} (\mathbf{S}(A_i) \otimes \mathrm{S}(A, A_i)) \to A_i(x) \tag{9}$$

$$\mathcal{S}_{\mathrm{C}} = \{A \in L^X \mid A = \mathrm{C}(A)\} \tag{10}$$

$$\mathcal{S}_{\mathbf{S}} = \{A \in L^X \mid \mathbf{S}(A) = 1\}, \tag{11}$$

$$\mathbf{S}_{\mathrm{C}}(A) = E(A, \mathrm{C}(A)) \tag{12}$$

$$\mathbf{S}_{\mathcal{S}}(A) = E(A, \mathrm{C}_{\mathcal{S}}(A)). \tag{13}$$

The situation is depicted in Fig. 1.

Theorem 7 ([2]) *Under the above notation,* $\mathrm{C}_{\mathcal{S}}$ *and* $\mathrm{C}_{\mathbf{S}}$ *are* $\mathbf{L}_K$*-closure operators,* $\mathcal{S}_{\mathrm{C}}$ *and* $\mathcal{S}_{\mathbf{S}}$ *are* $\mathbf{L}_K$*-closure systems,* $\mathbf{S}_{\mathrm{C}}$ *and* $\mathbf{S}_{\mathcal{S}}$ *are* $\mathbf{L}_K$*-closure* $\mathbf{L}$*-systems, and the diagram in Fig. 1 commutes.*

Each oriented path in the diagram of Fig. 1 defines a mapping (a mapping composed of the mappings represented by the arrows). Commutativity of the diagram in Fig. 1 says that any two mappings corresponding to oriented paths with common starting node and final node are equal. Particularly, we have that the mappings defined by (8)–(13) are pairwise inverse (i.e. we have $\mathrm{C} = \mathrm{C}_{\mathcal{S}_{\mathrm{C}}}$ and $\mathrm{C} = \mathrm{C}_{\mathbf{S}_{\mathrm{C}}}$, $\mathcal{S} = \mathcal{S}_{\mathrm{C}_{\mathcal{S}}}$ and $\mathcal{S} = \mathcal{S}_{\mathbf{S}_{\mathcal{S}}}$, and $\mathbf{S} = \mathbf{S}_{\mathrm{C}_{\mathbf{S}}}$ and $\mathbf{S} = \mathbf{S}_{\mathcal{S}_{\mathbf{S}}}$); furthermore, we have $\mathrm{C} = \mathrm{C}_{\mathcal{S}_{\mathbf{S}_{\mathrm{C}}}}$ etc.

Definition 8 *An* $\mathbf{L}_K$-Galois connection *(*fuzzy Galois connection*) between the sets* X *and* Y *is a pair* $\langle{}^\uparrow, {}^\downarrow\rangle$ *of mappings* ${}^\uparrow : L^X \to L^Y$, ${}^\downarrow : L^Y \to L^X$, *satisfying*

$$\begin{aligned}
\mathrm{S}(A_1, A_2) &\leq \mathrm{S}(A_2^\uparrow, A_1^\uparrow) \quad \textit{whenever } \mathrm{S}(A_1, A_2) \in K && (14)\\
\mathrm{S}(B_1, B_2) &\leq \mathrm{S}(B_2^\downarrow, B_1^\downarrow) \quad \textit{whenever } \mathrm{S}(B_1, B_2) \in K && (15)\\
A &\subseteq (A^\uparrow)^\downarrow && (16)\\
B &\subseteq (B^\downarrow)^\uparrow\ . && (17)
\end{aligned}$$

for every $A, A_1, A_2 \in L^X$, $B, B_1, B_2 \in L^Y$.

If $K = L$ then we again omit the subscript K. Note also that an $\mathbf{L}_K$-Galois connection between X and Y forms a Galois connection between the complete lattices $\langle L^X, \subseteq\rangle$ and $\langle L^Y, \subseteq\rangle$ [10, 21].

Remark Note that Galois connections between sets [10, 21] are just $\mathbf{L}$-Galois connections for $\mathbf{L} = \mathbf{2}$.

We will need the following results.

Theorem 9 ([7]) *Let* C *be an* $\mathbf{L}$*-closure operator, and* $Y = \{\mathrm{C}(A) \mid A \in L^X\}$. *Then the pair of mappings* ${}^{\uparrow_{\mathrm{C}}} : L^X \to L^Y$, ${}^{\downarrow_{\mathrm{C}}} : L^Y \to L^X$ *defined for* $A \in L^X$, $B \in L^Y$ *and* $x \in X$, $A' \in Y$ *by*

$$\begin{aligned}
A^{\uparrow_{\mathrm{C}}}(A') &= \mathrm{S}(A, A')\\
B^{\downarrow_{\mathrm{C}}}(x) &= \bigwedge_{A \in Y} B(A) \to A(x)
\end{aligned}$$

forms an $\mathbf{L}$*-Galois connection such that* $C = {}^{\uparrow_{\mathrm{C}}\downarrow_{\mathrm{C}}}$.

Theorem 10 ([2]) *For a binary* $\mathbf{L}$*-relation* $I \in L^{X\times Y}$ *denote* ${}^{\uparrow_I} : L^X \to L^Y$ *and* ${}^{\downarrow_I} : L^Y \to L^X$ *the mappings defined for* $A \in L^X$, $B \in L^Y$ *by*

$$\begin{aligned}
A^{\uparrow_I}(y) &= \bigwedge_{x \in X} (A(x) \to I(x, y)) && (18)\\
B^{\downarrow_I}(x) &= \bigwedge_{y \in Y} (B(y) \to I(x, y)). && (19)
\end{aligned}$$

For an $\mathbf{L}$*-Galois connection* $\langle{}^\uparrow, {}^\downarrow\rangle$ *between* X *and* Y *denote* $I_{\langle\uparrow,\downarrow\rangle}$ *the binary* $\mathbf{L}$*-relation* $I \in L^{X\times Y}$ *defined for* $x \in X$, $y \in Y$ *by* $I(x,y) = \{{}^1/x\}^\uparrow(y)\,(= \{{}^1/y\}^\downarrow(x))$. *Then* $\langle{}^{\uparrow_I}, {}^{\downarrow_I}\rangle$ *is an* $\mathbf{L}$*-Galois connection and it holds*

$$\langle{}^\uparrow, {}^\downarrow\rangle = \langle{}^{\uparrow_{I_{\langle\uparrow,\downarrow\rangle}}}, {}^{\downarrow_{I_{\langle\uparrow,\downarrow\rangle}}}\rangle \quad \textit{and} \quad I = I_{\langle\uparrow_I, \downarrow_I\rangle}.$$

3 Induced relations: quasiorder and similarity

An $\mathbf{L}$-relation R on a set X is called

$$\begin{array}{rl} \text{reflexive if} & R(x,x) = 1 \\ \text{symmetric if} & R(x,y) = R(y,x) \\ K\text{-transitive if} & R(x,y) \otimes R(y,z) \leq R(x,z) \\ & \text{whenever } R(x,y) \in K \text{ and } R(y,z) \in K. \end{array}$$

An $\mathbf{L}_K$-*quasiorder* on X is an $\mathbf{L}$-relation on X that is reflexive and K-transitive. An $\mathbf{L}_K$-*similarity* (or $\mathbf{L}_K$-equivalence) on X is an $\mathbf{L}$-relation on X that is reflexive, symmetric, and K-transitive.

Remark (1) Clearly, putting $\mathbf{L} = \mathbf{2}$ we get the usual (bivalent) notion of a quasiorder and an equivalence relation (no matter what K). Thus, the notions of $\mathbf{L}_K$-quasiorder and $\mathbf{L}_K$-similarity are generalizations of of the bivalent notions.

(2) If K is interpreted as the set of sufficiently high (designated) truth values, then K-transitivity means: "if the facts that $\langle x,y\rangle$ belongs to R and $\langle y,z\rangle$ belongs to R are sufficiently true then $\langle x,z\rangle$ also belongs to R (and is sufficiently true in case K is a filter)". For $K = L$ (in which case we omit the subscript K) we get the usual notions of $\mathbf{L}$-quasiorder and $\mathbf{L}$-similarity (called usually fuzzy quasiorder and fuzzy similarity).

Induced L-relations on X

Theorem 11 *Let* C *be an* $\mathbf{L}_K$*-closure operator on* X*. Then the relation* Q_{C} *in* X *defined by*

$$Q_{\mathrm{C}}(x,y) = \mathrm{C}(\{\,{}^{1}/x\})(y)$$

is an $\mathbf{L}_K$*-quasiorder.*

Proof. Since $\{\,{}^{1}/x\} \subseteq \mathrm{C}(\{\,{}^{1}/x\})$, reflexivity follows by

$$Q(x,x) = \mathrm{C}(\{\,{}^{1}/x\})(x) = 1.$$

K-transitivity: Let $Q_{\mathrm{C}}(x,y) \in K$, $Q_{\mathrm{C}}(y,z) \in K$. We have to show $Q_{\mathrm{C}}(x,y) \otimes Q_{\mathrm{C}}(y,z) \leq Q_{\mathrm{C}}(x,z)$. By adjointness and by definition of Q_{C} we thus have to show that

$$\mathrm{C}(\{\,{}^{1}/x\})(y) \leq \mathrm{C}(\{\,{}^{1}/y\})(z) \to \mathrm{C}(\{\,{}^{1}/x\})(z)$$

whenever $\mathrm{C}(\{\,{}^{1}/x\})(y)$, $\mathrm{C}(\{\,{}^{1}/y\})(z) \in K$. We have $\mathrm{S}(\{\,{}^{\mathrm{C}(\{\,{}^{1}/x\})(y)}/\}, \mathrm{C}(\{\,{}^{1}/x\})) = 1$, therefore, by (3) and using $\mathrm{C}(\mathrm{C}(\{\,{}^{1}/x\})) = \mathrm{C}(\{\,{}^{1}/x\})$, also $\mathrm{S}(\mathrm{C}(\{\,{}^{\mathrm{C}(\{\,{}^{1}/x\})(y)}/\}), \mathrm{C}(\{\,{}^{1}/x\})) = 1$. Furthermore, by $\mathrm{C}(\{\,{}^{1}/x\})(y) \in K$ and (3),

$$\begin{aligned} &\mathrm{C}(\{\,{}^{1}/x\})(y) = \mathrm{S}(\{\,{}^{1}/y\}, \{\,{}^{\mathrm{C}(\{\,{}^{1}/x\})(y)}/y\}) \leq \\ \leq\ &\mathrm{S}(\mathrm{C}(\{\,{}^{1}/y\}), \mathrm{C}(\{\,{}^{\mathrm{C}(\{\,{}^{1}/x\})(y)}/y\})). \end{aligned}$$

Therefore,

$$\begin{aligned}
& \mathrm{C}(\{{}^{1}/x\})(y) \leq \mathrm{S}(\mathrm{C}(\{{}^{1}/y\}), \mathrm{C}(\{{}^{\mathrm{C}(\{{}^{1}/x\})(y)}/y\})) = \\
= \;& \mathrm{S}(\mathrm{C}(\{{}^{1}/y\}), \mathrm{C}(\{{}^{\mathrm{C}(\{{}^{1}/x\})(y)}/y\})) \otimes \mathrm{S}(\mathrm{C}(\{{}^{\mathrm{C}(\{{}^{1}/x\})(y)}/\}), \mathrm{C}(\{{}^{1}/x\})) \leq \\
\leq \;& \mathrm{S}(\mathrm{C}(\{{}^{1}/y\}), \mathrm{C}(\{{}^{1}/x\})) \leq \mathrm{C}(\{{}^{1}/y\})(z) \to \mathrm{C}(\{{}^{1}/x\})(z).
\end{aligned}$$

The proof is complete. □

Remark Note that Q_{C} actually satisfies a stronger condition than K-transitivity. Namely, as it follows from the proof of Theorem 11, $Q_{\mathrm{C}}(x,y) \otimes Q_{\mathrm{C}}(y,z) \leq Q_{\mathrm{C}}(x,z)$ whenever $Q_{\mathrm{C}}(x,y) \in K$. This property is typical for Pavelka style fuzzy logic (see [22] and also [18]): Take X to be the set of all formulas; put $K = \{1\}$; let C be the operator of syntactic consequence, i.e. for an $\mathbf{L}$-set A of formulas and a formula $x \in X$, let $\mathrm{C}(A)(x)$ be the degree of provability of x from A. One easily verifies that C satisfies the above condition stronger than K-transitivity. On the other hand, C does not satisfy the in a sense symmetric condition, i.e. it is not true that if $Q_{\mathrm{C}}(y,z) \in K$ then $Q_{\mathrm{C}}(x,y) \otimes Q_{\mathrm{C}}(y,z) \leq Q_{\mathrm{C}}(x,z)$.

$Q_{\mathrm{C}}(x,y)$ is naturally interpreted as the truth degree to which y belongs to the closure of a singleton containing x. One might wonder what is the relationship between Q_{C} and $Q_{\mathcal{S}_{\mathrm{C}}}$ defined by

$$Q_{\mathcal{S}_{\mathrm{C}}}(x,y) = \bigwedge_{A \in \mathcal{S}_{\mathrm{C}}, A(x) \in K} A(x) \to A(y),$$

i.e. the truth degree to which it holds that whenever it is sufficiently true that x belongs to some closure then y belongs to that closure as well.

Theorem 12 *For any $\mathbf{L}_K$-closure operator* C *we have $Q_{\mathrm{C}} = Q_{\mathcal{S}_{\mathrm{C}}}$. Therefore, $Q_{\mathcal{S}_{\mathrm{C}}}$ is an $\mathbf{L}_K$-quasiorder.*

Proof. On the one hand, $\mathrm{C}(\{{}^{1}/x\})(x) = 1 \in K$ yields

$$\begin{aligned}
& Q_{\mathcal{S}_{\mathrm{C}}}(x,y) = \bigwedge_{A \in \mathcal{S}_{\mathrm{C}}, A(x) \in K} A(x) \to A(y) \leq \\
\leq \;& \mathrm{C}(\{{}^{1}/x\})(x) \to \mathrm{C}(\{{}^{1}/x\})(y) = 1 \to \mathrm{C}(\{{}^{1}/x\})(y) = \\
= \;& \mathrm{C}(\{{}^{1}/x\})(y) = Q_{\mathrm{C}}(x,y).
\end{aligned}$$

On the other hand, $Q_{\mathrm{C}}(x,y) \leq Q_{\mathcal{S}_{\mathrm{C}}}(x,y)$ is true iff for each $A \in \mathcal{S}_{\mathrm{C}}$ such that $A(x) \in K$ we have $\mathrm{C}(\{{}^{1}/x\})(y) \leq A(x) \to A(y)$. Applying adjointness twice, the last inequality is equivalent to $A(x) \leq \mathrm{C}(\{{}^{1}/x\})(y) \to A(y)$ which is true. Indeed, since $A(x) \in K$, (3) gives

$$\begin{aligned}
& A(x) = 1 \to A(x) = \{{}^{1}/x\}(x) \to A(x) = \mathrm{S}(\{{}^{1}/x\}, A) \leq \\
\leq \;& \mathrm{S}(\mathrm{C}(\{{}^{1}/x\}), \mathrm{C}(A)) = \mathrm{S}(\mathrm{C}(\{{}^{1}/x\}), A) \leq \mathrm{C}(\{{}^{1}/x\}(y) \to A(y).
\end{aligned}$$

□

If K is, moreover, a filter in $\mathbf{L}$, the fact that Q_{C} is an $\mathbf{L}_K$-quasiorder follows from Theorem 12 and the following statement.

Lemma 13 *Let K be an filter in $\mathbf{L}$, $\mathcal{S} = \{A_i \in L^X \mid i \in I\}$ be a system of $\mathbf{L}$-sets. Then the $\mathbf{L}$-relation $Q_{\mathcal{S}}$ on X given by*

$$Q_{\mathcal{S}}(x,y) = \bigwedge_{i\in I, A_i(x)\in K} A_i(x) \to A_i(y)$$

is an $\mathbf{L}_K$-quasiorder.

Proof. Reflexivity follows from the fact that $A_i(x) \to A_i(x) = 1$ and from $\bigwedge \emptyset = 1$. K-transitivity: By definition, we have to show that if

$$\bigwedge_{i\in I, A_i(x)\in K} A_i(x) \to A_i(y) \in K$$

and

$$\bigwedge_{i\in I, A_i(x)\in K} A_i(y) \to A_i(z) \in K$$

then

$$\begin{aligned} &(\bigwedge_{i\in I, A_i(x)\in K} A_i(x) \to A_i(y)) \otimes (\bigwedge_{i\in I, A_i(y)\in K} A_i(y) \to A_i(z)) \leq \\ \leq \;& (\bigwedge_{i\in I, A_i(x)\in K} A_i(x) \to A_i(z)), \end{aligned}$$

i.e. to show that for each $i \in I$ such that $A_i(x) \in K$ we have

$$\begin{aligned} &A_i(x) \otimes (\bigwedge_{i\in I, A_i(x)\in K} A_i(x) \to A_i(y)) \otimes (\bigwedge_{i\in I, A_i(y)\in K} A_i(y) \to A_i(z)) \leq \\ \leq \;& A_i(z). \end{aligned}$$

This inequality is true. Indeed, $\bigwedge_{i\in I, A_i(x)\in K} A_i(x) \to A_i(y) \in K$ and $\bigwedge_{i\in I, A_i(x)\in K} A_i(x) \to A_i(y) \leq A_i(x) \to A_i(y)$ gives $A_i(x) \to A_i(y) \in K$. As also $A_i(x) \in K$, we have $A_i(x) \otimes (A_i(x) \to A_i(y)) \in K$. From $A_i(x) \otimes (A_i(x) \to A_i(y)) \leq A_i(y)$ we thus have $A_i(y) \in K$. Therefore, we conclude

$$\begin{aligned} &A_i(x) \otimes (\bigwedge_{i\in I, A_i(x)\in K} A_i(x) \to A_i(y)) \otimes (\bigwedge_{i\in I, A_i(x)\in K} A_i(y) \to A_i(z)) \leq \\ \leq \;& A_i(x) \otimes (A_i(x) \to A_i(y)) \otimes (A_i(y) \to A_i(z)) \leq A_i(z), \end{aligned}$$

completing the proof. □

Indeed, putting $\mathcal{S} = \mathcal{S}_{\mathrm{C}}$, Lemma 13 yields that $Q_{\mathcal{S}_{\mathrm{C}}}$ is an $\mathbf{L}_K$-quasiorder. Theorem 12 then completes the argument.

Remark (1) A closer look at the proof of Lemma 13 shows that like Q_{C}, $Q_{\mathcal{S}}$ satisfies a stronger form of K-transitivity: $Q_{\mathcal{S}}(x,y)\otimes Q_{\mathcal{S}}(y,z) \leq Q_{\mathcal{S}}(x,z)$ whenever $Q_{\mathcal{S}}(x,y) \in K$.

(2) The assumption of Lemma 13 that K be closed w.r.t. $\otimes$ is essential. As a counterexample, consider $I = \{i,j\}$, $X = \{x,y,z\}$, $L = [0,1]$ equipped with Łukasiewicz structure, $A_i(x) = 0.9$, $A_i(y) = 0.8$, $A_i(z) = 0.7$, $A_j(x) = A_j(y) = A_j(z) = 1$. Taking $K = [0.9, 1]$ which is an $\leq$-filter not closed w.r.t. $\otimes$, we have $Q_{\mathcal{S}}(x,y) \otimes Q_{\mathcal{S}}(y,z) = 0.9 \otimes 1 = 0.9 \not\leq 0.8 = Q_{\mathcal{S}}(x,z)$.

It can be easily seen that every $\mathbf{L}_K$-quasiorder on X induces an $\mathbf{L}_K$-similarity E_Q on X by putting $E_Q(x,y) = Q(x,y)\wedge Q(y,x)$. Therefore, for any $\mathbf{L}_K$-closure operator C on X, the $\mathbf{L}$-relation E_{C} on X defined by

$$E_{\mathrm{C}}(x,y) = Q_{\mathrm{C}}(x,y) \wedge Q_{\mathrm{C}}(y,x)$$

is an $\mathbf{L}_K$-similarity on X. We say that an $\mathbf{L}_K$-similarity E on X is compatible with $A \in L^X$ if $A(x) \otimes E(x,y) \leq A(y)$ holds for any $x, y \in X$ such that $A(x) \in K$. The condition of compatibility translates verbally to "if it is sufficiently true that x belongs to A and if x and y are similar then y belongs to A as well".

Theorem 14 *Let* C *be an* $\mathbf{L}_K$*-closure operator on* X*. Then* E_{C} *is the largest* $\mathbf{L}_K$*-similarity on* X *that is compatible w.r.t. every* C*-closed* $\mathbf{L}$*-set (i.e. w.r.t. every* $A \in \mathcal{S}_{\mathrm{C}}$*).*

Proof. The fact that E_{C} is an $\mathbf{L}_K$-similarity on X was established in the above paragraph. Let $x, y \in X$, $A \in \mathcal{S}_{\mathrm{C}}$, $A(x) \in K$. By Theorem 12,

$$\begin{aligned} E_{\mathrm{C}}(x,y) &= Q_{\mathcal{S}_{\mathrm{C}}}(x,y) \wedge Q_{\mathcal{S}_{\mathrm{C}}}(y,x) \leq Q_{\mathcal{S}_{\mathrm{C}}}(x,y) = \\ &= \bigwedge_{A\in\mathcal{S}_{\mathrm{C}}, A(x)\in K} A(x) \to A(y) \leq A(x) \to A(y), \end{aligned}$$

i.e. $A(x) \otimes E_{\mathrm{C}}(x,y) \leq A(y)$ by adjointness. We proved that E_{C} is compatible w.r.t. any $A \in \mathcal{S}_{\mathrm{C}}$.

Let E be an $\mathbf{L}_K$-similarity that is compatible w.r.t. any $A \in \mathcal{S}_{\mathrm{C}}$. Take $x, y \in X$ and an $A \in \mathcal{S}_{\mathrm{C}}$ such that $A(x) \in K$. Compatibility of E yields $A(x) \otimes E(x,y) \leq A(y)$, i.e. $E(x,y) \leq A(x) \to A(y)$. Since x, y, and A were chosen arbitrarily, we conclude

$$E(x,y) \leq \bigwedge_{A\in\mathcal{S}_{\mathrm{C}}, A(x)\in K} A(x) \to A(y) \leq Q_{\mathcal{S}_{\mathrm{C}}}(x,y).$$

Due to the symmetry of E we finally have

$$E(x,y) \leq Q_{\mathcal{S}_{\mathrm{C}}}(x,y) \wedge Q_{\mathcal{S}_{\mathrm{C}}}(y,x) = E_{\mathrm{C}}(x,y)$$

proving that E_{C} is the largest $\mathbf{L}_K$-similarity compatible with all $A \in \mathcal{S}_{\mathrm{C}}$. $\square$

Factorization of $\mathbf{L}_K$-closure systems by similarity For an $\mathbf{L}_K$-closure operator on X, $\mathcal{S}_{\mathrm{C}}$ is a complete lattice w.r.t. $\subseteq$. This fact follows directly from the fact that $\mathcal{S}_{\mathrm{C}}$ is closed w.r.t. arbitrary intersections. For various reasons (e.g. for computational ones), it might not be desirable to distinguish the particular $\mathbf{L}$-sets in X. Rather, it can be advantageous to treat $\mathbf{L}$-sets which are similar in terms of membership degrees of elements of X as if they were the same, i.e. one might desire to perform a kind of abstraction by factorization w.r.t. to a suitable similarity defined on $\mathbf{L}$-sets. A suitable $\mathbf{L}$-similarity relation is described in the following assertion (see e.g. [4]).

Lemma 15 *The* $\mathbf{L}$*-relation* E *on* X *defined by*

$$E(A,B) = \bigwedge_{x\in X} A(x) \leftrightarrow B(x)$$

is an $\mathbf{L}$*-similarity on* X*.*

For $A,B \in L^X$, $E(A,B)$ is the truth degree to which it is true that for any $x \in X$, x belongs to A iff x belongs to B. The first observation states that sufficiently high similarity between $\mathbf{L}$-sets is preserved by $\mathbf{L}_K$-closure operators.

Theorem 16 *For an* $\mathbf{L}_K$*-closure operator* C *on* X *and* $A,B \in L^X$ *we have* $E(A,B) \le E(\mathrm{C}(A),\mathrm{C}(B))$ *whenever* $E(A,B) \in K$*.*

Proof. It is easy to see that $E(A,B) = \mathrm{S}(A,B) \wedge \mathrm{S}(B,A)$. Therefore, $E(A,B) \in K$ yields $\mathrm{S}(A,B) \in K$ and $\mathrm{S}(B,A) \in K$. Applying (3) we get $\mathrm{S}(A,B) \le \mathrm{S}(\mathrm{C}(A),\mathrm{C}(B))$ and $\mathrm{S}(B,A) \le \mathrm{S}(\mathrm{C}(B),\mathrm{C}(A))$, and thus $E(A,B) = \mathrm{S}(A,B) \wedge \mathrm{S}(B,A) \le \mathrm{S}(\mathrm{C}(A),\mathrm{C}(B)) \wedge \mathrm{S}(\mathrm{C}(B),\mathrm{C}(A)) = E(\mathrm{C}(A),\mathrm{C}(B))$. □

Intuitively, the complete lattice $\mathcal{S}_{\mathrm{C}}$ can be simplified by putting similar closed $\mathbf{L}$-sets together, i.e. putting together $\mathbf{L}$-sets A and B for which $E(A,B)$ is high. Putting the $\mathbf{L}$-sets together should be compatible w.r.t. the complete lattice structure on $\mathcal{S}_{\mathrm{C}}$. Recall that for any $a \in L$, an a-cut of E is a (bivalent) relation ${}^aE$ on X defined by $\langle x,y\rangle \in {}^aE$ iff $a \le E(x,y)$. It is easy to see that ${}^aE$ is always a tolerance on X (i.e. a reflexive and symmetric relation on X). If $\otimes = \wedge$, ${}^aE$ is, moreover, transitive, i.e. an equivalence relation. In general, ${}^aE$ is not transitive. Factorization of a structure by a compatible tolerance relation is, in general, not possible (one needs transitivity so that operations on the factor set can be defined). Surprisingly, Czédli [13] showed a way to factorize lattices by compatible tolerances (for factorization of complete lattices by tolerances see [26]). We now recall the necessary concepts: Let T be a tolerance relation on a support V of a complete lattice $\mathbf{V} = \langle V,\le\rangle$. T is called compatible if it is preserved under arbitrary infima and suprema, i.e. if $\langle u_i,v_i\rangle \in T$ $(i \in I)$ implies $\langle \bigwedge_{i\in I} u_i, \bigwedge_{i\in I} v_i\rangle \in T$ and $\langle \bigvee_{i\in I} u_i, \bigvee_{i\in I} v_i\rangle \in T$. For $v \in V$, denote $v_T = \bigwedge_{\langle v,v'\rangle\in T} v'$ and $v^T = \bigvee_{\langle v,v'\rangle\in T} v'$, and call each set of the form $[v]_T = [v_T, v^T] = \{v' \in V \mid v_T \le v' \le v^T\}$ a block of T. Denote $V/T = \{[v]_T \mid v \in V\}$ the set of all blocks of T and call it the factor set of V by

T. Introduce a relation $\leq_T$ defined on V/T by $[v]_T \leq [v']_T$ iff $\bigwedge[v]_T \leq \bigwedge[v']_T$ (or, equivalently, iff $\bigvee[v]_T \leq \bigvee[v']_T$). The following assertion follows from [26].

Theorem 17 *Let* C *be an* $\mathbf{L}_K$*-closure operator on* X*,* T *be a compatible tolerance relation on* $\langle \mathcal{S}_{\mathrm{C}}, \subseteq \rangle$*. (1)* $\mathcal{S}_{\mathrm{C}}/T$ *is the set of all maximal blocks of* T*, i.e.* $\mathcal{S}_{\mathrm{C}}/T = \{B \subseteq \mathcal{S}_{\mathrm{C}} \mid B \times B \subseteq T \ \&\ ((\forall B' \subset B) B' \times B' \not\subseteq T)\}$*. (2)* $\langle \mathcal{S}_{\mathrm{C}}/T, \subseteq_T \rangle$ *is a complete lattice (factor lattice) where infima and suprema are given by*

$$\bigwedge_{i\in I}[A_i]_T = [(\bigcap_{i\in I} A_i)^T]_T \quad \textit{and} \quad \bigvee_{i\in I}[A_i]_T = [\bigvee_{i\in I} A_i]_T.$$

One may easily verify that if T is, moreover, transitive (i.e. a complete congruence on $\mathbf{V}$), then $\langle \mathcal{S}_{\mathrm{C}}, \subseteq_T \rangle$ is the well-known factor lattice.

To show that the a-cuts ${}^aE$ can be used to factorize $\mathcal{S}_{\mathrm{C}}$ by the above described procedure, we need to verify that ${}^aE$ is compatible w.r.t. $\subseteq$.

Lemma 18 *Let* C *be an* $\mathbf{L}_K$*-closure operator on* X*. For any* $a \in K$*,* ${}^aE$ *is a compatible tolerance relation on the complete lattice* $\langle \mathcal{S}_{\mathrm{C}}, \subseteq \rangle$*.*

Proof. We show that ${}^aE$ is compatible both with infima and suprema, i.e. we show that $\langle A_i, B_i \rangle \in {}^aE$ $(i \in I)$ implies both $\langle \bigwedge_{i\in I} A_i, \bigwedge_{i\in I} B_i \rangle \in {}^aE$ and $\langle \bigvee_{i\in I} A_i, \bigvee_{i\in I} B_i \rangle \in {}^aE$.

Infima: Suppose $\langle A_i, B_i \rangle \in {}^aE$, i.e. $a \leq \bigwedge_{x\in X}(A_i(x) \leftrightarrow B_i(x))$ $(i \in I)$. We have to show

$$a \leq \bigwedge_{x\in X} (\bigwedge_{i\in I} A_i(x) \leftrightarrow \bigwedge_{i\in I} B_i(x)),$$

i.e. to show that for each $i \in I$ we have

$$a \leq (\bigwedge_{i\in I} A_i(x) \leftrightarrow \bigwedge_{i\in I} B_i(x)).$$

The last inequality is true iff both $a \leq (\bigwedge_{i\in I} A_i(x) \rightarrow \bigwedge_{i\in I} B_i(x))$ and $a \leq (\bigwedge_{i\in I} B_i(x) \rightarrow \bigwedge_{i\in I} A_i(x))$ are valid. Due to symmetry we verify only $a \leq (\bigwedge_{i\in I} A_i(x) \rightarrow \bigwedge_{i\in I} B_i(x))$ which is equivalent to $a \otimes \bigwedge_{i\in I} A_i(x) \leq \bigwedge_{i\in I} B_i(x)$. This inequality is true. Indeed, by assumption, $\langle A_i, B_i \rangle \in {}^aE$, i.e. $a \leq \bigwedge_{x\in X}(A_i(x) \leftrightarrow B_i(x))$, from which it follows $a \otimes A_i(x) \leq B_i(x)$ for any $i \in I$, $x \in X$. We therefore have

$$\begin{aligned} & a \otimes \bigwedge_{i\in I} A_i(x) \leq \\ \leq \quad & \bigwedge_{i\in I}(a \otimes A_i(x)) \leq \bigwedge_{i\in I} B_i(x). \end{aligned}$$

Suprema: We have to show $a \leq E(\bigvee_{i\in I} A_i, \bigvee_{i\in I} B_i)$, i.e. $a \leq E(\mathrm{C}(\bigcup_{i\in I} A_i), \mathrm{C}(\bigcup_{i\in I} B_i))$. First, observe that $(*)$ $a \leq E(\bigcup_{i\in I} A_i, \bigcup_{i\in I} B_i)$: Indeed, the inequality is true iff both $a \leq \mathrm{S}(\bigcup_{i\in I} A_i, \bigcup_{i\in I} B_i)$ and $a \leq \mathrm{S}(\bigcup_{i\in I} B_i, \bigcup_{i\in I} A_i)$ hold. Because of symmetry, we verify only the former one: by definition, $a \leq \mathrm{S}(\bigcup_{i\in I} A_i, \bigcup_{i\in I} B_i)$ is true iff $a \leq (\bigcup_{i\in I} A_i)(x) \to (\bigcup_{i\in I} B_i)(x)$, i.e. iff $a \otimes (\bigcup_{i\in I} A_i)(x) \leq (\bigcup_{i\in I} B_i)(x)$ holds for each $x \in X$. By assumption,

$$a \otimes (\bigcup_{i\in I} A_i)(x) = \bigvee_{i\in I} (a \otimes A_i(x)) \leq \bigvee_{i\in I} B_i(x) = (\bigcup_{i\in I} B_i)(x)$$

establishing $(*)$.

Now, $a \leq E(\bigcup_{i\in I} A_i, \bigcup_{i\in I} B_i)$ implies both $a \leq \mathrm{S}(\bigcup_{i\in I} A_i, \bigcup_{i\in I} B_i)$ and $a \leq \mathrm{S}(\bigcup_{i\in I} B_i, \bigcup_{i\in I} A_i)$. Since $a \in K$, (3) implies $a \leq \mathrm{S}(\mathrm{C}(\bigcup_{i\in I} A_i), \mathrm{C}(\bigcup_{i\in I} B_i))$ and $a \leq \mathrm{S}(\mathrm{C}(\bigcup_{i\in I} B_i), \mathrm{C}(\bigcup_{i\in I} A_i))$, i.e.

$$a \leq E(\mathrm{C}(\bigcup_{i\in I} A_i), \mathrm{C}(\bigcup_{i\in I} B_i)) = E(\bigvee_{i\in I} A_i, \bigvee_{i\in I} B_i)$$

completing the proof. $\square$

Remark (1) Note that the tolerance relation ${}^aE$ used to factorize $\mathcal{S}_{\mathrm{C}}$ need not to be supplied from the outside. It is determined by selecting an appropriate $a \in K$.

(2) The role of $a \in K$ is to control the granularity of the factorization: since $a \leq b$ implies ${}^bE \subseteq {}^aE$, the rule is "the bigger a, the finer the factorization". Clearly, for the extreme cases of a, i.e. $a = 0$ (note that ${}^0E = \mathcal{S}_{\mathrm{C}} \times \mathcal{S}_{\mathrm{C}}$ which is always a compatible relation on $\mathcal{S}_{\mathrm{C}}$) and $a = 1$ we obtain $\mathcal{S}_{\mathrm{C}}/{}^0E$ which is a one-element lattice and $\mathcal{S}_{\mathrm{C}}/{}^1E$ which is a lattice isomorphic to $\mathcal{S}_{\mathrm{C}}$.

(3) Note also that the fact $a \in K$ is not needed in the proof of compatibility with infima in $\mathcal{S}_{\mathrm{C}}$.

Factorization of $\mathbf{L}_K$-closure systems by decrease of logical precision We now mention another way to factorize a system $\mathcal{S}_{\mathrm{C}}$of all closed $\mathbf{L}$-sets of an $\mathbf{L}$-closure operator C. Let thus C be an $\mathbf{L}_1$-closure operator on X. It may be the case that one does not need to distinguish close truth values of $\mathbf{L}_1$. Formally, a kind of a factorization of $\mathbf{L}_1$ is taking place. Assume therefore that there is a complete homomorphism h of $\mathbf{L}_1$ onto $\mathbf{L}_2$ (i.e. a homomorphism which preserves arbitrary infima and suprema). h induces a mapping $h^* : L_1^X \to L_2^X$ such that for $A \in L_1^X$, $h^*(A) \in L_2^X$ is defined by

$$(h^*(A))(x) = h(A(x))$$

for any $x \in X$. Since there is no danger of misunderstanding, we write simply h instead of h^*.

One may consider h as representing a decrease of logical precision (see [6]): Several truth values from L_1 may collapse into one truth value from L_2; instead of $A \in L_1^X$ one may consider $h^*(A) \in L_2^X$ which may be easier to work with, yet sufficiently granular.

Furthermore, h induces a mapping $h(\mathrm{C}) : L_2^X \to L_2^X$ as follows: for $B \in L_2^X$ take any $A \in L_1^X$ such that $h(A) = B$ (such an A always exists due to the fact that h is surjective) and define

$$(h(\mathrm{C}))(B) = h(\mathrm{C}(A)).$$

Theorem 19 *Let* C *be an* $\mathbf{L}_1$*-closure operator in* X*,* h *be a complete homomorphism of* $\mathbf{L}_1$ *onto* $\mathbf{L}_2$*. Then* $h(\mathrm{C})$ *is an* $\mathbf{L}_2$*-closure operator in* X *and* $A \mapsto h(A)$ *is a complete lattice homomorphism of* $\mathcal{S}_\mathrm{C}$ *onto* $\mathcal{S}_{h(\mathrm{C})}$*.*

Proof. First, we show that $h(\mathrm{C})$ is defined correctly. In order to do it, take any $A_1, A_2 \in L_1^X$ such that $h(A_1) = h(A_2)$. We need to show $h(\mathrm{C}(A_1)) = h(\mathrm{C}(A_2))$. It is easy to show that $h(E(A_1, A_2)) = 1$. Since $E(A_1, A_2) \leq E(\mathrm{C}(A_1), \mathrm{C}(A_2))$, monotonicity of h gives $h(E(\mathrm{C}(A_1), \mathrm{C}(A_2)))$. As $h(E(\mathrm{C}(A_1), \mathrm{C}(A_2))) = E(h(\mathrm{C}(A_1)), h(\mathrm{C}(A_2)))$, we infer $E(h(\mathrm{C}(A_1)), h(\mathrm{C}(A_2))) = 1$, whence $h(\mathrm{C}(A_1)) = h(\mathrm{C}(A_2))$. Therefore, $h(\mathrm{C})$ is defined correctly.

Let $\langle{}^{\uparrow_\mathrm{C}}, {}^{\downarrow_\mathrm{C}}\rangle$ be the $\mathbf{L}$-Galois connection between X and $\mathcal{S}_\mathrm{C}$ of Theorem 9. Consider an $\mathbf{L}_1$-relation I between X and $\mathcal{S}_\mathrm{C}$ defined for $x \in X$ and $A \in \mathcal{S}_\mathrm{C}$ by

$$I(x, A) = A(x).$$

Let $\langle{}^{\uparrow_I}, {}^{\downarrow_I}\rangle$ be the $\mathbf{L}_1$-Galois connection between X and $\mathcal{S}_\mathrm{C}$ according to Theorem 10. It is immediate that $\langle{}^{\uparrow_\mathrm{C}}, {}^{\downarrow_\mathrm{C}}\rangle = \langle{}^{\uparrow_I}, {}^{\downarrow_I}\rangle$. We will show that $h(\mathrm{C}) = {}^{\uparrow_{h(I)}\downarrow_{h(I)}}$, i.e. $h(\mathrm{C})$ is a composition of ${}^{\uparrow_{h(I)}}$ and ${}^{\downarrow_{h(I)}}$. To this end, take any $A \in L_1^X$. Using repeatedly the fact that h is a complete homomorphism, we get

$$h(A)^{\uparrow_{h(I)}\downarrow_{h(I)}} = \cdots = h(A^{\uparrow_I\downarrow_I}) = h(\mathrm{C}(A)) = (h(\mathrm{C}))(A)$$

proving $h(\mathrm{C}) = {}^{\uparrow_{h(I)}\downarrow_{h(I)}}$. By Theorem 9, $h(\mathrm{C})$ is an $\mathbf{L}_2$-closure operator in X.

Finally, by [6], $\langle A, A^{\uparrow_I}\rangle \mapsto \langle h(A), h(A^{\uparrow_I})\rangle$ is a complete lattice homomorphism of $\mathcal{B}_I = \{\langle A, A^{\uparrow_I}\rangle \mid A \in \mathcal{S}_\mathrm{C}\}$ onto $\mathcal{B}_{h(I)} = \{\langle A, A^{\uparrow_{h(I)}}\rangle \mid A \in \mathcal{S}_{h(\mathrm{C})}\}$ (with the lattice order $\leq$ on $\mathcal{B}_I$ given by $\langle A, A^{\uparrow_I}\rangle \leq \langle B, B^{\uparrow_I}\rangle$ iff $A \subseteq B$, similarly for $\mathcal{B}_{h(I)}$). The fact that $A \mapsto h(A)$ is a complete lattice homomorphism of $\mathcal{S}_\mathrm{C}$ onto $\mathcal{S}_{h(\mathrm{C})}$ now directly follows by observing that $\mathcal{B}_I$ is isomorphic to $\mathcal{S}_\mathrm{C}$, and $\mathcal{B}_{h(I)}$ is isomorphic to $\mathcal{S}_{h(\mathrm{C})}$. □

$\mathcal{S}_{h(\mathrm{C})}$ is therefore a homomorphic image of $\mathcal{S}_\mathrm{C}$, i.e. a factor lattice of $\mathcal{S}_\mathrm{C}$.

Remark The fact that $h(\mathrm{C})$ is an $\mathbf{L}_2$-closure operator can be proved directly (without reference to Theorem 9) as follows: For any $A \in L_1^X$ we have $A \subseteq \mathrm{C}(A)$, whence $h(A) \subseteq h(\mathrm{C}(A)) = (h(\mathrm{C}))(A)$ by monotonicity of h, proving (2). Furthermore, $\mathrm{S}(A_1, A_2) \leq \mathrm{S}(\mathrm{C}(A_1), \mathrm{C}(A_2))$ implies

$$\begin{aligned} & \mathrm{S}(h(A_1), h(A_2)) = h(\mathrm{S}(A_1, A_2)) \leq h(\mathrm{S}(\mathrm{C}(A_1), \mathrm{C}(A_2))) = \\ = \ & \mathrm{S}(h(\mathrm{C}(A_1)), h(\mathrm{C}(A_2))) = \mathrm{S}((h(\mathrm{C}))(A_1), (h(\mathrm{C}))(A_2)) \end{aligned}$$

proving (3). Finally,

$$\begin{aligned} & (h(\mathrm{C}))(h(A)) = h(\mathrm{C}(A)) = h(\mathrm{C}(\mathrm{C}(A))) = \\ = \quad & (h(\mathrm{C}))(h(\mathrm{C}(A))) = (h(\mathrm{C}))((h(\mathrm{C}))(h(A))) \end{aligned}$$

proving (4).

4 Representation by 2-closure operators

We show that there is a natural one-to-one correspondence between $\mathbf{L}_{\{1\}}$-closure operators on X and special closure operators on $X \times L$. Call a subset $A \subseteq X \times L$ *(*$\mathbf{L}$*-set)-representative* if (1) for each $x \in X$ it holds $\langle x, a\rangle \in A$ and $b \leq a$ implies $\langle x, b\rangle \in A$, and (2) for each $x \in X$ the set $\{a \in L \mid \langle x, a\rangle \in A\}$ has the greatest element.

For any $\mathbf{L}$-set $A \in L^X$ put

$$\lfloor A \rfloor = \{\langle x, a\rangle \in X \times L \mid a \leq A(x)\}. \tag{20}$$

For any set $A \subseteq X \times L$ put

$$\lceil A \rceil = \{\langle x, a\rangle \in X \times L \mid a = \bigvee_{\langle x, b\rangle \in A} b\}. \tag{21}$$

We have immediately the following result.

Lemma 20 *Let $A \in L^X$ be an $\mathbf{L}$-set, $A' \subseteq X \times L$ be a representative set. Then (1) $\lfloor A \rfloor \subseteq X \times L$ is an representative set, (2) $\lceil A' \rceil$ is an $\mathbf{L}$-set such that (3) $A = \lceil \lfloor A \rfloor \rceil$, $A' = \lfloor \lceil A' \rceil \rfloor$.*

Definition 21 *A $\mathbf{2}$-closure operator* D *on $X \times L$ is called* commutative w.r.t. $\lfloor\lceil\ \rceil\rfloor$ *if*

$$\lfloor \lceil \mathrm{D}(A) \rceil \rfloor = \mathrm{D}(A) = \mathrm{D}(\lfloor \lceil A \rceil \rfloor) \tag{22}$$

holds for each $A \in X \times L$.

Remark It is easy to verify that (22) holds iff both $\lfloor \lceil \mathrm{D}(A) \rceil \rfloor \subseteq \mathrm{D}(A)$ and $\mathrm{D}(\lfloor \lceil A \rceil \rfloor) \subseteq \mathrm{D}(A)$ hold.

For an operator $\mathrm{D} : X \times L \to X \times L$ define an operator $\mathrm{C}_{\mathrm{D}} : L^X \to L^X$ by

$$\mathrm{C}_{\mathrm{D}}(A) = \lceil \mathrm{D}(\lfloor \mathrm{A} \rfloor) \rceil$$

for $A \in L^X$. For an operator $\mathrm{C} : L^X \to L^X$ define an operator $\mathrm{D}_{\mathrm{C}} : X \times L \to X \times L$ by

$$\mathrm{D}_{\mathrm{C}}(A) = \lfloor \mathrm{C}(\lceil \mathrm{A} \rceil) \rfloor$$

for $A \in X \times L$.

Theorem 22 *Let* C *be an* $\mathbf{L}_{\{1\}}$*-closure operator on* X *and* D *be a* $\mathbf{2}$*-closure operator on* $X \times L$ *which is commutative w.r.t.* $\lfloor \lceil\ \rceil \rfloor$*. Then (1)* D_{C} *is a* $\mathbf{2}$*-closure operator on* X *which is commutative w.r.t.* $\lfloor \lceil\ \rceil \rfloor$*, (2)* C_{D} *is an* $\mathbf{L}_{\{1\}}$*-closure operator on* X*, and (3)* $\mathrm{C} = \mathrm{C}_{\mathrm{D}_{\mathrm{C}}}$ *and* $\mathrm{D} = \mathrm{D}_{\mathrm{C}_{\mathrm{D}}}$*.*

Proof. (1) Let $A, A_1, A_2 \subseteq X \times L$. We have

$$A \subseteq \lfloor \lceil A \rceil \rfloor \subseteq \lfloor \mathrm{C}(\lceil A \rceil) \rfloor = \mathrm{D}_{\mathrm{C}}(A),$$

proving extensionality of D_{C}. If $A_1 \subseteq A_2$ then clearly $\mathrm{S}(\lceil A_1 \rceil, \lceil A_2 \rceil) = 1$, hence $\mathrm{S}(\mathrm{C}(\lceil A_1 \rceil), \mathrm{C}(\lceil A_2 \rceil)) = 1$, so

$$\mathrm{D}_{\mathrm{C}}(A_1) = \lfloor \mathrm{C}(\lceil A_1 \rceil) \rfloor \subseteq \lfloor \mathrm{C}(\lceil A_2 \rceil) \rfloor = \mathrm{D}_{\mathrm{C}}(A_2),$$

proving monotonicity of D_{C}.

$$\mathrm{D}_{\mathrm{C}}(\mathrm{D}_{\mathrm{C}}(A)) = \lfloor \mathrm{C}(\lceil \lfloor \mathrm{C}(\lceil A \rceil) \rfloor \rceil) \rfloor = \lfloor \mathrm{C}(\mathrm{C}(\lceil A \rceil)) \rfloor = \lfloor \mathrm{C}(\lceil A \rceil) \rfloor = \mathrm{D}_{\mathrm{C}}(A),$$

proving idempotency. Finally,

$$\lfloor \lceil \mathrm{D}_{\mathrm{C}}(A) \rceil \rfloor = \lfloor \lceil \lfloor \mathrm{C}(\lceil A \rceil) \rfloor \rceil \rfloor = \lfloor \mathrm{C}(\lceil A \rceil) \rfloor = \mathrm{D}_{\mathrm{C}}(A),$$

and

$$\mathrm{D}_{\mathrm{C}}(\lfloor \lceil A \rceil \rfloor) = \lfloor \mathrm{C}(\lceil \lfloor \lceil A \rceil \rfloor \rceil) \rfloor = \lfloor \mathrm{C}(\lceil A \rceil) \rfloor = \mathrm{D}_{\mathrm{C}}(A),$$

verifying commutativity of D_{C}.

(2) Let $A, A_1, A_2 \in L^X$. We have $A = \lceil \lfloor A \rfloor \rceil \subseteq \lceil \mathrm{D}(\lfloor A \rfloor) \rceil = \mathrm{C}_{\mathrm{D}}(A)$, thus C_{D} is extensional. If $\mathrm{S}(A_1, A_2) = 1$ then $\lfloor A_1 \rfloor \subseteq \lfloor A_2 \rfloor$, therefore $\mathrm{D}(\lfloor A_1 \rfloor) \subseteq \mathrm{D}(\lfloor A_2 \rfloor)$ and thus

$$\mathrm{C}_{\mathrm{D}}(A_1) = \lceil \mathrm{D}(\lfloor A_1 \rfloor) \rceil \subseteq \lceil \mathrm{D}(\lfloor A_2 \rfloor) \rceil = \mathrm{C}_{\mathrm{D}}(A_2),$$

monotonicity of C_{D}. Using commutativity we further get

$$\mathrm{C}_{\mathrm{D}}(\mathrm{C}_{\mathrm{D}}(A)) = \lceil \mathrm{D}(\lfloor \lceil \mathrm{D}(\lfloor A \rfloor) \rceil \rfloor) \rceil = \lceil \mathrm{D}(\mathrm{D}(\lfloor A \rfloor)) \rceil = \lceil \mathrm{D}(\lfloor A \rfloor) \rceil = \mathrm{C}_{\mathrm{D}}(A),$$

idempotency of C_{D}.

(3) For any $A \in L^X$ we have $\mathrm{C}_{\mathrm{D}_{\mathrm{C}}}(A) = \lceil \lfloor \mathrm{C}(\lceil \lfloor A \rfloor \rceil) \rfloor \rceil = \mathrm{C}(A)$. For any $A \subseteq X \times L$ we have by commutativity of D that $\mathrm{D}_{\mathrm{C}_{\mathrm{D}}}(A) = \lfloor \lceil \mathrm{D}(\lfloor \lceil A \rceil \rfloor) \rceil \rfloor = \mathrm{D}(A)$. $\square$

Remark Note that commutativity of D is essential in the foregoing proposition (a counterexample is easy to get).

5 Some examples

In this section we introduce two further properties of $\mathbf{L}_K$-closure operators and show some examples. Call an $\mathbf{L}$-set $A \in L^X$ *finite* if $\{x \in X \mid A(x) > 0\}$ is a finite set.

Definition 23 *An* $\mathbf{L}_K$ *closure operator* C *on* X *is called* compact *(finitary, or algebraic) if*

$$\mathrm{C}(A) = \bigcup\{\mathrm{C}(B) \mid B \in L^X, B \subseteq A, B \text{ \textit{is finite} }\}$$

holds for each $A \in L^X$.

Remark For $\mathbf{L} = \mathbf{2}$ we get the compact closure operators.

Definition 24 *An* $\mathbf{L}_K$*-closure operator* C *on* X *is called* topologic *if it satisfies*

$$\mathrm{C}(A \cup B) = \mathrm{C}(A) \cup \mathrm{C}(B)$$

for any $A, B \in L^X$.

Remark For $\mathbf{L} = \mathbf{2}$, topologic $\mathbf{L}_K$-closure operators are just closure operators of topologic spaces. The corresponding system $\mathcal{S}_{\mathrm{C}}$ consists of the closed sets of the topology.

Fuzzy subalgebras Let $\mathbf{A} = \langle A, F\rangle$ be an algebra, i.e. A is a nonempty set and F is a system of operations on A. An $\mathbf{L}$-set $B \in L^A$ is called an $\mathbf{L}$-subalgebra of $\mathbf{A}$ if for each $f : A^n \to A$ of F and every $a_1, \ldots, a_n \in A$, it holds

$$B(a_1) \otimes \cdots \otimes B(a_n) \leq B(f(a_1, \ldots, a_n)).$$

Denote by $\mathbf{L}\text{-Sub}\,\mathbf{A}$ the set of all $\mathbf{L}$-subalgebras of $\mathbf{A}$.

Remark For $\otimes = \wedge$, $\mathbf{L}$-subalgebras and their systems are introduced and investigated in [8]. Note that $\mathbf{2}$-subalgebras coincide with the usual subalgebras.

Theorem 25 *For any algebra* $\mathbf{A} = \langle A, F\rangle$, $\mathbf{L}\text{-Sub}\,\mathbf{A}$ *is an* $\mathbf{L}_{\{1\}}$*-closure system and the corresponding operator* $\mathrm{C} = \mathrm{C}_{\mathbf{L}\text{-Sub}\,\mathbf{A}}$ *is an algebraic* $\mathbf{L}_{\{1\}}$*-closure operator. Moreover, if* $\otimes = \wedge$ *(i.e.* $\mathbf{L}$ *is an algebra of intuitionistic logic (Heyting algebra)),* C *is an* $\mathbf{L}$*-closure operator.*

Proof. It is easy to see that **L**-Sub **A** is closed under arbitrary intersections, hence **L**-Sub **A** is an $\mathbf{L}_{\{1\}}$-closure system and the corresponding C is an $\mathbf{L}_{\{1\}}$-closure operator. It remains to verify the compactness of C. To this end, let $B \in L^A$ and put

$$[B](a) = \bigvee\{B(a_1)^{|x_1|_t} \otimes \cdots \otimes B(a_n)^{|x_n|_t} \mid t \in T_n,\, a_i \in A, t(a_1,\ldots,a_n) = a\}$$

where $\alpha^k = \alpha \otimes \cdots \otimes \alpha$ (k-times), T_i denotes the set of all i-ary terms of the type of **A**, and $|x_i|_t$ denotes the number of occurrences of the variable x_i in t. It is a matter of routine to prove by induction over rank of the term t (defined by $\mathrm{ran}\,(x_i) = 0$ and $\mathrm{ran}\,(f(t_1,\ldots,t_n)) = 1 + \max\{\mathrm{ran}\,(t_1),\ldots,\mathrm{ran}\,(t_n)\}$) that $[B] = \mathrm{C}(B)$ which implies the compactness of C.

Let $\otimes = \wedge$. We have to prove $\mathrm{S}(B_1, B_2) \leq \mathrm{S}([B_1],[B_2])$ for every $B_1, B_2 \in L^A$ which holds iff for each $a \in A$ we have $[B_1](a) \wedge \mathrm{S}(B_1,B_2) \leq [B_2](a)$. Since $\wedge$ is idempotent we have

$$[B_1](a) \wedge \mathrm{S}(B_1,B_2) = \tag{23}$$
$$= \bigvee_{t\in T_n, t(a_1,\ldots,a_n)=a} (B_1(a_1) \wedge \cdots \wedge B_1(a_n)) \wedge \mathrm{S}(B_1,B_2) = \tag{24}$$
$$= \bigvee_{t\in T_n, t(a_1,\ldots,a_n)=a} (B_1(a_1) \wedge \mathrm{S}(B_1,B_2) \wedge \cdots \wedge B_1(a_n) \wedge \mathrm{S}(B_1,B_2)) \leq \tag{25}$$
$$\leq \bigvee_{t\in T_n, t(a_1,\ldots,a_n)=a} (B_2(a_1) \wedge \cdots \wedge B_2(a_n)) = [B_2](a). \tag{26}$$

□

Remark Note that in general, $\mathrm{C} = \mathrm{C}_{\mathbf{L}\text{-Sub}\,\mathbf{A}}$ is not an **L**-closure system. As a counterexample consider $L = \{0, \frac{1}{2}, 1\}$ with a Łukasiewicz structure, and a four-element lattice with the support $A = \{a,b,c,d\}$ with the least element a, the greatest element d, and two mutually incomparable elements b and c, i.e. its Hasse diagram is a 45^o-rotated square. Let $B_1, B_2 \in L^A$ be given by $B_1(a) = 0$, $B_1(b) = B_1(c) = B_1(d) = 1$, $B_2(a) = 0$, $B_2(b) = B_2(c) = B_2(d) = \frac{1}{2}$. Clearly, B_2 itself is an **L**-subalgebra of **A**. On the other hand, $[B_1](a) = 1$ since $B_1(b) \otimes B_1(c) == 1 \leq [B_1](b \wedge c) = [B_1](a)$. We therefore have $\mathrm{S}(B_1,B_2) = \frac{1}{2} \not\leq 0 = [B_1](a) \to [B_2](a) = \mathrm{S}([B_1],[B_2])$, i.e. $[\;] = \mathrm{C}$ is not an **L**-closure operator.

Fuzzy relational closures Let R be an **L**-relation on the set X, i.e. $R \in L^{X\times X}$. By a reflexive (symmetric, transitive) closure of R it is meant the least **L**-relation on X which is itself reflexive (symmetric, transitive) and contains R. The reflexive, symmetric, and transitive closure of R is denoted by R^r, R^s, and R^t, respectively. Recall that R is reflexive if $R(x,x) = 1$, symmetric if $R(x,y) = R(y,x)$, and transitive if $R(x,y) \otimes R(y,z) \leq R(x,z)$.

It is immediate that $R^r = R \cup_{x\in X} \{{}^1/\langle x,x\rangle\}$, $R^s = R \cup R^{-1}$ where $R^{-1}(x,y) = R(y,x)$. Since $R \subseteq R^r$, $R \subseteq R^s$, $\mathrm{S}(R,S) \leq \mathrm{S}(R^r,S^r)$, $\mathrm{S}(R,S) \leq$

$\mathrm{S}(R^s, S^s)$ (both of the inequalities are easy to verify), and $R^r = R^{rr}$, $R^s = R^{ss}$, we conclude that both ^{r} and ^{s} are $\mathbf{L}$-closure operators on $X \times X$. Moreover, by the above description of ^{r} and ^{s} we conclude that both of them are compact as well as topologic.

To show that $R^t = \bigcup_{i=1}^{\infty} R^i = R \cup R \circ R \cup R \circ R \circ R \cup \cdots$ (where $(R \circ S)(x,y) = \bigvee_{z \in X}(R(x,z) \otimes S(z,y))$ it is enough to observe that $R \subseteq \bigcup_{i=1}^{\infty} R^i$; if $R \subseteq S$ and S is transitive then $\bigcup_{i=1}^{\infty} R^i \subseteq S$; and that $\bigcup_{i=1}^{\infty} R^i$ is transitive. Since the two former are evident, we only verify the last condition. $\bigcup_{i=1}^{\infty} R^i$ is transitive iff

$$(\bigcup_{i=1}^{\infty} R^i)(x,y) \circ (\bigcup_{j=1}^{\infty} R^j)(y,z) = (\bigvee_{i=1}^{\infty} R^i(x,y)) \circ (\bigvee_{j=1}^{\infty} R^j(y,z)) =$$
$$= \bigvee_{i=1}^{\infty} \bigvee_{j=1}^{\infty} R^i(x,y) \circ R^j(y,z) \leq \bigvee_{k=1}^{\infty} R^k(x,z)$$

which holds iff for every i,j we have $R^i(x,y) \circ R^j(y,z) \leq \bigvee_{k=1}^{\infty} R^k(x,z)$. The last statement is true because $R^i(x,y) \circ R^j(y,z) \leq R^{i+j}(x,z)$. It is easy to see that for $K = \{1\}$, the conditions (2)–(4) are satisfied, hence ^{t} is an $\mathbf{L}_K$-closure operator on $X \times X$. In general, ^{t} is not an $\mathbf{L}$-closure operator (consider $L = \{0, \frac{1}{2}, 1\}$ with Łukasiewicz structure, $X = \{a,b,c\}$, $R(a,b) = R(b,c) = 1$, $S(a,b) = S(b,c) = \frac{1}{2}$, and $R(x,y) = S(x,y) = 0$ otherwise). ^{t} is compact since

$$R^t(x,y) = \bigvee_{i=1}^{\infty} R^i(x,y) =$$
$$= \bigvee_{i=1}^{\infty} \bigvee_{x=z_1,z_2,\ldots,z_{i+1}=y} R(z_1,z_2) \otimes \cdots \otimes R(z_i,z_{i+1}) =$$
$$= \bigvee_{i=1}^{\infty} \bigvee_{x=z_1,z_2,\ldots,z_{i+1}=y} \{ {}^{R(z_1,z_2)}\!/\langle z_1,z_2\rangle, \ldots, {}^{R(z_i,z_{i+1})}\!/\langle z_i,z_{i+1}\rangle \}^t(x,y).$$

As it is well-known from the classical case ($L = \{0,1\}$), ^{t} is not topologic.

Remark An easy inspection shows that if $\mathbf{L}$ is a Heyting algebra ($\otimes = \wedge$) then ^{t} is even an $\mathbf{L}$-closure operator on $X \times X$.

Fuzzy concept lattices By Port-Royal logic [1], a concept is determined by its extent (the collection of all objects which fall under the concept) and its intent (the collection of all attributes which fall under the concept). For instance, the extent of the concept DOG is the collection of all dogs while its intent is the collection of all attributes common to dogs (like "to be a mammal", "to bark" etc.). Port-Royal theory of concepts has been formalized and developed into a logico-algebraical theory of conceptual data analysis and knowledge representation by Wille et al. [25, 14]. The theory is known as formal concept analysis or theory of concept lattices. The first approach to generalize formal concept analysis from the point of view of fuzzy logic is [11]. Later on, a general approach to the study of concept lattices from the point of view of fuzzy logic has been pursued, independently, by Pollandt [23] and the present author (see e.g. [2, 3, 4, 6, 5]). The theory goes as follows: Let X and Y be non-empty sets interpreted as the set of objects and the set of attributes, respectively, I be an **L**-relation between X and Y. The triple $\langle X, Y, I\rangle$ is called a (formal) **L**-context. A (formal) **L**-concept in $\langle X, Y, I\rangle$ is a pair $\langle A, B\rangle \in L^X \times L^Y$ (i.e. A is an **L**-set of attributes, B is an **L**-set of attributes) such that B is the **L**-set of all attributes common to all objects from A, and A is the **L**-set of all objects sharing all the attributes from B. These verbal conditions translate formally as follows: Let ${}^{\uparrow_I} : L^X \to L^Y$ and ${}^{\downarrow_I} : L^Y \to L^X$ be defined by (18) and (19). Then $\langle A, B\rangle \in L^X \times L^Y$ is an **L**-concept in $\langle X, Y, I\rangle$ iff $A^{\uparrow} = B$ and $B^{\downarrow} = A$ (one easily verifies that the verbal conditions are expressed exactly by formulas (18) and (19)). The set $\mathcal{B}(X, Y, I) = \{\langle A, B\rangle \in L^X \times L^Y \mid A^{\uparrow} = B,\ B^{\downarrow} = A\}$ equipped with the partial order $\leq$ defined by

$$\langle A_1, B_1\rangle \leq \langle A_2, B_2\rangle \quad \text{iff } A_1 \subseteq A_2 \quad (\text{iff } B_2 \subseteq B_1)$$

is called the **L**-concept lattice (fuzzy concept lattice) determined by $\langle X, Y, I\rangle$. Fuzzy concept lattice (which is in fact a complete lattice) is the basic derived structure which reveals the conceptual knowledge present in (the input data) $\langle X, Y, I\rangle$ (for more information on the principles of conceptual data analysis see [14]).

Putting in other words, Theorem 10 says that $\langle{}^{\uparrow_I}, {}^{\downarrow_I}\rangle$ forms a representative form of **L**-Galois connection. Theorem 9 implies that the composite mappings ${}^{\uparrow_I\downarrow_I} : L^X \to L^X$ and ${}^{\downarrow_I\uparrow_I} : L^Y \to L^X$ are **L**-closure operators on X and Y, respectively, and that the sets $\{A \in L^X \mid A = A^{\uparrow_I\downarrow_I}\}$ and $\{B \in L^X \mid B = B^{\uparrow_I\downarrow_I}\}$ are dually isomorphic **L**-closure systems. Moreover, Theorem 9 implies that each **L**-closure operator on X (**L**-closure system in X) can be viewed as being of the form ${}^{\uparrow_I\downarrow_I}$ (or ${}^{\downarrow_I\uparrow_I}$) (as the set of all extents (or intents) of an **L**-concept lattice) for some **L**-context $\langle X, Y, I\rangle$.

We now show an application of so-called Main theorem of **L**-concept lattices (see [3]) to provide a characterization of lattices of fixed points of **L**-closure operators. To this end, recall that given a complete lattice $\mathbf{V} = \langle V, \leq\rangle$, a subset $K \subseteq V$ is called infimally (supremally) dense in $\mathbf{V}$ if each $v \in V$ is an infimum (supremum) of some subset of K.

Theorem 26 *A complete lattice* $\mathbf{V} = \langle V, \leq\rangle$ *is isomorphic to* $\langle \mathcal{S}_{\mathrm{C}}, \subseteq\rangle$ *for an* **L***-closure operator* C *in* X *iff there are mappings* $\gamma : X \times L \to V$ *and* $\mu : \mathcal{S}_{\mathrm{C}} \times L \to V$ *such that* $\gamma(X, L)$ *is supremally dense in* $\mathbf{V}$*,* $\mu(\mathcal{S}_{\mathrm{C}}, L)$ *is infimally dense in* $\mathbf{V}$*, and* $a \otimes b \leq A(x)$ *is equivalent to* $\gamma(x, a) \leq \mu(A, b)$ *for any* $a, b \in L$*,* $x \in X$*,* $A \in \mathcal{S}_{\mathrm{C}}$*.*

Proof. The proof follows immediately from [3, Theorem 7], Theorem 9, and Theorem 10. □

Since $\mathcal{B}(X,Y,I)$ is isomorphic to $\mathcal{S}_{\mathrm{C}}$ for $\mathrm{C} = {}^{\uparrow_I \downarrow_I}$, the above two factorization procedures for $\mathcal{S}_{\mathrm{C}}$ yield immediately factorization procedures for fuzzy concept lattices (see also [4, 6]).

6 Fuzzy closure operators and consequence relations

The concept of a closure operator is an important one from the point of view of logic. Typically, a closure operator C in a given logical calculus arises as follows: for a given set collection A of formulas, the closure $\mathrm{C}(A)$ is defined to be the collection of all formulas provable from A. A detailed study of closure operators in the context of two-valued logic can be found in [27]. The situation is analogous in fuzzy logic. Namely, following the seminal work of Pavelka [22], provability degree of a formula from a fuzzy set of formulas is defined in fuzzy logic. Then, a fuzzy closure operator is naturally induced by a fuzzy logical calculus as follows: for a given fuzzy set A of formulas and a given formula φ, the degree to which φ belongs to the closure $\mathrm{C}(A)$ of A is defined to be the provability degree of φ from A. For a special structure of truth values (namely, for $L = [0,1]$ equipped with min as the connective modeling conjunction, i.e. the standard Gödel algebra [18]), fuzzy closure operators and fuzzy consequence relations have been studied by Chakraborty (see e.g. [12]) and Gerla (see e.g. [15]). However, the study of general fuzzy closure operators and its relations to fuzzy logic is still an open goal (a paper on this topic is in preparation).

Our aim in this section is to present a general result on the relationship between fuzzy closure operators and fuzzy consequence relations. First, we show that each binary **L**-relation between L^X (the set of all **L**-sets in a given set X) and X induces in a natural way a fuzzy closure system (and the corresponding fuzzy closure operator). For an **L**-relation R between L^X and X, and a subset $K \subseteq L$, we say that an **L**-set $A \in L^X$ is R_K-closed if for any $B \in L^X$ and each $x \in X$ we have

$$\mathrm{S}(B,A) \otimes R(B,x) \leq A(x)$$

whenever $\mathrm{S}(B,A) \in K$.

Lemma 27 *For any $R \in L^{L^X \times X}$ and $K \subseteq L$, the set $\mathcal{S}_R$ of all* **L***-sets in X that are R_K-closed forms an $\mathbf{L}_K$-closure system.*

Proof. By definition, we have to show that $\mathcal{S}_R$ is closed w.r.t. S_K-intersections, i.e. we have to show that for any $A \in L^X$, $(\bigcap_{B \in \mathcal{S}_R, \mathrm{S}(A,B) \in K} \mathrm{S}(A,B) \to B)$ is R_K-closed. Take any $C \in L^X$ such that $\mathrm{S}(C, \bigcap_{B \in \mathcal{S}_R, \mathrm{S}(A,B) \in K} \mathrm{S}(A,B) \to B) \in K$. We have to show

$$\begin{aligned} & \mathrm{S}(C, \bigcap_{B \in \mathcal{S}_R, \mathrm{S}(A,B) \in K} \mathrm{S}(A,B) \to B) \otimes R(C,x) \leq \\ \leq & \bigwedge_{B \in \mathcal{S}_R, \mathrm{S}(A,B) \in K} \mathrm{S}(A,B) \to B(x) \end{aligned}$$

is true for any $x \in X$. The last inequality holds iff for any $B \in L^X$ such that $\mathrm{S}(A,B) \in K$ we have

$$\mathrm{S}(A,B) \otimes \mathrm{S}(C, \bigcap_{B \in \mathcal{S}_R, \mathrm{S}(A,B) \in K} \mathrm{S}(A,B) \to B) \otimes R(C,x) \leq B(x)$$

which is clearly true provided both $\mathrm{S}(A,B) \otimes \mathrm{S}(C, \bigcap_{B \in \mathcal{S}_R, \mathrm{S}(A,B) \in K} \mathrm{S}(A,B) \to B) \leq \mathrm{S}(C,B)$ and $\mathrm{S}(C,B) \in K$ are valid. We have

$$\begin{aligned}
& \mathrm{S}(A,B) \otimes \mathrm{S}(C, \bigcap_{B \in \mathcal{S}_R, \mathrm{S}(A,B) \in K} \mathrm{S}(A,B) \to B) \leq \\
\leq\ & \mathrm{S}(A,B) \otimes \mathrm{S}(C, \mathrm{S}(A,B) \to B) = \\
=\ & \mathrm{S}(A,B) \otimes \bigwedge_{y \in X} (C(y) \to (\mathrm{S}(A,B) \to B(y))) = \\
=\ & \mathrm{S}(A,B) \otimes \bigwedge_{y \in X} (\mathrm{S}(A,B) \to (C(y) \to B(y))) \leq \\
\leq\ & \bigwedge_{y \in X} (\mathrm{S}(A,B) \otimes (\mathrm{S}(A,B) \to (C(y) \to B(y)))) \leq \\
\leq\ & \bigwedge_{y \in X} C(y) \to B(y) = \mathrm{S}(C,B).
\end{aligned}$$

Since $\mathrm{S}(A,B) \in K$ and $\mathrm{S}(C, \bigcap_{B \in \mathcal{S}_R, \mathrm{S}(A,B) \in K} \mathrm{S}(A,B) \to B) \in K$, we conclude $\mathrm{S}(C,B) \in K$. $\square$

In fact, each $\mathbf{L}_K$-closure system is induced in the way described in Lemma 27 by its corresponding $\mathbf{L}_K$-closure operator: Each $\mathbf{L}_K$-closure operator C on X induces an $\mathbf{L}$-relation $R_{\mathrm{C}} \in L^{L^X \times X}$ by

$$R_{\mathrm{C}}(A,x) = \mathrm{C}(A)(x).$$

Now, applying Lemma 27 to R_{C} we get the $\mathbf{L}_K$-closure system corresponding to C:

Lemma 28 *For an $\mathbf{L}_K$-closure operator C on X we have $\mathcal{S}_{\mathrm{C}} = \mathcal{S}_{R_{\mathrm{C}}}$.*

Proof. Let $A \in \mathcal{S}_{\mathrm{C}}$. We show that A is R_K-closed. Let $\mathrm{S}(B,A) \in K$. Then, by (3), $\mathrm{S}(B,A) \leq \mathrm{S}(\mathrm{C}(B), \mathrm{C}(A))$, i.e. for each $x \in X$ we have

$$\mathrm{S}(B,A) \otimes R_{\mathrm{C}}(B,x) = \mathrm{S}(B,A) \otimes \mathrm{C}(B)(x) \leq \mathrm{C}(A)(x) = A(x),$$

whence A is R_K-closed. Conversely, if A is R_K-closed then for any B such that $\mathrm{S}(B,A) \in K$ we have $\mathrm{S}(B,A) \otimes R_{\mathrm{C}}(B,x) \leq A(x)$. Putting $B = A$ and considering $\mathrm{S}(A,A) = 1 \in K$, we conclude

$$\mathrm{S}(B,A) \otimes R_{\mathrm{C}}(B,x) = 1 \otimes R_{\mathrm{C}}(A,x) = \mathrm{C}(A)(x) \leq A(x),$$

thus $A \in \mathcal{S}_{\mathrm{C}}$. $\square$

Recall (see e.g. [15]) that a (bivalent) relation $\vdash$ between 2^X and X is called a consequence relation in (a given set) X if (i) $X \vdash \varphi$ holds for each $\varphi \in X$; (ii) $X \vdash \varphi$ and $X \subseteq Y$ imply $Y \vdash \varphi$; and (iii) $X \cup Y \vdash \varphi$ and $X \vdash \psi$ for any $\psi \in Y$ imply $X \vdash \varphi$.

Definition 29 *An* $\mathbf{L}$*-relation* $\vdash$ *between* L^X *and* X *is called an* $\mathbf{L}_K$*-consequence relation provided it satisfies*

(i) $A(\varphi) \leq (A \vdash \varphi)$

(ii) $(A \vdash \varphi) \otimes \mathrm{S}(A, B) \;\leq\; (B \vdash \varphi)$ *whenever* $(B \vdash \varphi) \in K$

(iii) $(\bigwedge_{\varphi \in X} B(\psi) \;\to\; (A \;\vdash\; \psi)) \otimes ((A \cup B) \;\vdash\; \varphi) \;\;\leq\;\; (A \;\vdash\; \varphi)$ *whenever* $(\bigwedge_{\varphi \in X} B(\psi) \to (A \vdash \psi)) \in K$

for any $A, B \in L^X$, $\varphi \in X$.

Remark Note that condition (iii) may be equivalently replaced by (iii'):

$$(\bigwedge_{\varphi \in X} B(\psi) \to (A \vdash \psi)) \otimes (B \vdash \varphi) \;\leq\; (A \vdash \varphi)$$

whenever $(\bigwedge_{\varphi \in X} B(\psi) \to (A \vdash \psi)) \in K$. Indeed, $(B \vdash \varphi) \leq ((A \cup B) \vdash \varphi)$ by (ii), whence (iii) implies (iii'). Conversely, suppose (iii') and take $B' = A \cup B$. By $A(\psi) \leq (A \vdash \psi)$ we have

$$\begin{aligned}
&\bigwedge_{\psi \in X} ((A \cup B)(\psi) \to (A \vdash \psi)) = \\
= &\bigwedge_{\psi \in X} (A(\psi) \to (A \vdash \psi)) \wedge \bigwedge_{\psi \in X} (B(\psi) \to (A \vdash \psi)) = \\
= &\bigwedge_{\psi \in X} (B(\psi) \to (A \vdash \psi)).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&(\bigwedge_{\psi \in X} (B(\psi) \to (A \vdash \psi))) \otimes ((A \cup B) \vdash \varphi) = \\
= &(\bigwedge_{\psi \in X} ((A \cup B)(\psi) \to (A \vdash \psi))) \otimes ((A \cup B) \vdash \varphi) = \\
= &(\bigwedge_{\psi \in X} (B'(\psi) \to (A \vdash \psi))) \otimes (B' \vdash \varphi) \leq (A \vdash \varphi)
\end{aligned}$$

by (iii').

For a mapping $\mathrm{C} : L^X \to L^X$, define an $\mathbf{L}$-relation $\vdash_{\mathrm{C}}$ between L^X and X by

$$(A \vdash_{\mathrm{C}} \varphi) = \mathrm{C}(A)(\varphi).$$

For an $\mathbf{L}$-relation $\vdash$ between L^X and X define a mapping $\mathrm{C}_{\vdash} : L^X \to L^X$ by

$$\mathrm{C}_{\vdash}(A)(\varphi) = (A \vdash \varphi).$$

Theorem 30 *Let* $\mathrm{C} : L^X \to L^X$ *be a mapping,* $\vdash \in L^{L^X \times X}$ *be an* $\mathbf{L}$*-relation. Then (1)* C *is an* $\mathbf{L}_K$*-closure operator iff* $\vdash_{\mathrm{C}}$ *is an* $\mathbf{L}_K$*-consequence relation; (2)* $\vdash$ *is an* $\mathbf{L}_K$*-consequence relation iff* $\mathrm{C}_{\vdash}$ *is an* $\mathbf{L}_K$*-closure operator; (3)* $\mathrm{C} = \mathrm{C}_{\vdash_{\mathrm{C}}}$ *and* $\vdash = \vdash_{\mathrm{C}_{\vdash}}$.

Proof. Clearly, (3) is true. Therefore, it is sufficient to establish the "$\Rightarrow$"-parts of (1) and (2).

(1): We verify that $\vdash_{\mathrm{C}}$ is an $\mathbf{L}_K$-consequence operator. (i) and (ii) are direct consequences of (2) and (3). We now verify (iii'), a condition equivalent to (iii) (see Remark following Definition 29):

$$(\bigwedge_{\psi \in X} B(\psi) \to (A \vdash_{\mathrm{C}} \psi)) \otimes (B \vdash_{\mathrm{C}} \varphi) \leq (A \vdash_{\mathrm{C}} \varphi)$$

is by definition equivalent to

$$(\bigwedge_{\psi \in X} B(\psi) \to \mathrm{C}(A)(\psi)) \otimes \mathrm{C}(B)(\varphi) \leq \mathrm{C}(A)(\varphi)$$

which is true iff

$$\bigwedge_{\psi \in X} B(\psi) \to \mathrm{C}(A)(\psi) \leq \mathrm{C}(B)(\varphi) \to \mathrm{C}(A)(\varphi).$$

The last inequality is valid since $\bigwedge_{\psi \in X} B(\psi) \to \mathrm{C}(A)(\psi) \in K$ implies

$$\begin{aligned} & \bigwedge_{\psi \in X} B(\psi) \to \mathrm{C}(A)(\psi) = \mathrm{S}(B, \mathrm{C}(A)) \leq \\ \leq \quad & \mathrm{S}(\mathrm{C}(B), \mathrm{C}(\mathrm{C}(A))) \leq \mathrm{C}(B)(\varphi) \to \mathrm{C}(A)(\varphi). \end{aligned}$$

(2): We check that $\mathrm{C}_{\vdash}$ is an $\mathbf{L}_K$-closure operator. (2) and (3) are direct consequences of (i) and (ii). Putting $B = \mathrm{C}_{\vdash}(A)$, (iii') yields

$$\begin{aligned} & (\mathrm{C}_{\vdash}(\mathrm{C}_{\vdash}(A)))(\varphi) = 1 \otimes (\mathrm{C}_{\vdash}(A) \vdash \varphi) = \\ = \quad & \bigwedge_{\psi \in X} (B(\psi) \to (A \vdash \psi)) \otimes (B \vdash \varphi) \leq (A \vdash \varphi) = \mathrm{C}_{\vdash}(A)(\varphi). \end{aligned}$$

$\square$

Remark Theorem 30 thus establishes a one-to-one correspondence between fuzzy closure operators and fuzzy consequence relations. In [15], Gerla defines graded consequence relation in X as a fuzzy relation (Gerla takes $L = [0,1]$ and $\otimes = \min$) between $\mathbf{2}^X$ (the power set of X) and X. Gerla then establishes a one-to-one correspondence between graded consequence relations and special fuzzy closure operators (Gerla deals, in our terms, with $\mathbf{L}_{\{1\}}$-closure operators), i.e. not all fuzzy closure operators. As it can be easily seen, the difficulty is in condition (iii) of the definition of $\mathbf{L}_K$-consequence relation: particularly, the condition $(\bigwedge_{\varphi \in X} B(\psi) \to (A \vdash \psi)) \in K$ is missing in Gerla's definition. Instead, Gerla uses (iii"): $\bigwedge_{\psi \in B}(A \vdash \psi) \otimes ((A \cup B) \vdash \varphi) \leq (A \vdash \varphi)$ where $A, B \in 2^X$ (i.e. are A, B are subsets of X). Clearly, for $A, B \in 2^X$, (iii") is equivalent to $(\bigwedge_{\varphi \in X} B(\psi) \to (A \vdash \psi)) \otimes ((A \cup B) \vdash \varphi) \leq (A \vdash \varphi)$. Now, it is not true (as Gerla observes) that (iii") is satisfied by $\vdash_{\mathrm{C}}$ for any $\mathbf{L}_{\{1\}}$-closure operator C, i.e. (iii") is too strong and the above natural relations do not establish a one-to-one correspondence. Theorem 30 show a way to have a one-to-one correspondence, generalizing fully the bivalent case.

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