Fundamenta Informaticae 58 (2003) 1–13
IOS Press

Fuzzy Closure Operators Induced by Similarity

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Abstract. In fuzzy set theory, similarity phenomenon is approached using so-called fuzzy equivalence relations. An important role in fuzzy modeling is played by similarity-based closure (called also the extensional hull). Intuitively, the degree to which an element x belongs to a similarity-based closure of a fuzzy set A is the degree to which it is true that there is an element y in A which is similar to x. In this paper, we show a basic relationship between similarity-based closure and metric closure, and provide an axiomatic characterization of the operation of a similarity-based closure.

Keywords: similarity, fuzzy equivalence, fuzzy closure operator

1. Introduction

The concept of similarity and related concepts of distance, nearness, proximity, closeness etc. are among the basic concepts when modeling real-world phenomena. Of the most common approaches that allow us to quantify distance (or nearness) of objects of interest is the concept of a metric space. Fuzzy set theory offers another concept for modeling of similarity, so-called fuzzy equivalence. Briefly speaking, a fuzzy equivalence is a binary fuzzy relation E on a set (i.e. assigning to each pair $\langle x, y \rangle$ the truth degree $E(x, y)$ to which x and y are similar) which is reflexive, symmetric, and transitive. A given fuzzy equivalence E on a set X can be understood as an indistinguishability underlying the particular

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situation. From this point of view, it is natural to consider only those fuzzy sets A in X which satisfy a natural condition saying that if x belongs to A and if x and y are indistinguishable then y belongs to A as well (fuzzy sets satisfying this condition are called compatible with E). Only the compatible fuzzy sets respect the underlying indistinguishability. For a fuzzy set B in A, the smallest fuzzy set A in X containing B which is compatible with E is called the extensional hull of B. Another natural way to come to the concept of an extensional hull is the following. Let B represent a user-query in that $B(x)$ is the degree to which the element x is considered to satisfy the query. B may contain only a small number of "examples" specified by the user. Now, the user wants to get the collection (fuzzy set) A of all elements x for which there is some y in B which is similar to x, i.e. he or she wants to get all elements satisfying the query represented by the example B. It can be shown (and is well-known) that A is exactly the extensional hull of B.

The concepts of a fuzzy equivalence and that of an extensional hull of a fuzzy set are among the very important concepts having natural interpretation, interesting properties, and immediate applications, see e.g. [6, 9, 10, 11, 12, 15]. The aim of this paper is to investigate the concept of an extensional hull and to give its complete characterization in terms of so-called fuzzy closure operators [3]. Moreover, we discuss the relationship between the concept of the extensional hull (i.e. a similarity-based closure) and that of the metric-based closure.

In Section 2 we recall the necessary notions. The results of the paper and discussion is presented in Section 3.

2. Preliminaries

We recall necessary notions from fuzzy logic and fuzzy sets. We will use complete residuated lattices as the structures of truth values. Complete residuated lattices play a crucial role in fuzzy logic (see [9, 10, 11]). Being introduced in 1930s [17] as an abstraction in the study of ideal systems of rings, they have been proposed as a suitable structure of truth values by Goguen in [7, 8]. Recall that a complete residuated lattice is an algebra $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ such that $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice with the least element 0 and the greatest element 1; $\langle L, \otimes, 1\rangle$ is a commutative monoid, i.e. $\otimes$ is commutative, associative, and $x \otimes 1 = x$ holds for each $x \in L$; and $\otimes, \rightarrow$ form an adjoint pair, i.e. $x \otimes y \leq z$ iff $x \leq y \rightarrow z$ holds for all $x, y, z \in L$. $\otimes$ and $\rightarrow$ are called multiplication and residuum, respectively. All properties of complete residuated lattices used in this paper can be found in [10, 11]. The most studied and applied set of truth values is the real interval $[0, 1]$ with $a \wedge b = \min(a, b)$, $a \vee b = \max(a, b)$, and with three important pairs of adjoint operations: the Łukasiewicz one ($a \otimes b = \max(a+b-1, 0)$, $a \rightarrow b = \min(1-a+b, 1)$), Gödel one ($a \otimes b = \min(a, b)$, $a \rightarrow b = 1$ if $a \leq b$ and $= b$ otherwise), and product one ($a \otimes b = a \cdot b$, $a \rightarrow b = 1$ if $a \leq b$ and $= b/a$ otherwise); see [10] for their role in fuzzy logic. More generally, $\langle [0, 1], \min, \max, \otimes, \rightarrow, 0, 1\rangle$ is a complete residuated lattice on $[0, 1]$ iff $\otimes$ is a left-continuous t-norm [10] and $a \rightarrow b = \max\{z \mid a \otimes z \leq b\}$. Another important set of truth values is the set $\{a_0 = 0, a_1, \ldots, a_n = 1\}$ ($a_0 < \cdots < a_n$) with $\otimes$ given by $a_k \otimes a_l = a_{\max(k+l-n,0)}$ and the corresponding $\rightarrow$ given by $a_k \rightarrow a_l = a_{\min(n-k+l,n)}$. A special case of the latter algebras is the Boolean algebra $\mathbf{2}$ of classical logic with the support $2 = \{0, 1\}$. It may be easily verified that the only t-norm on $\{0, 1\}$ is the classical conjunction operation $\wedge$, i.e. $a \wedge b = 1$ iff $a = 1$ and $b = 1$, which implies that the only residuum operation is the classical implication operation $\rightarrow$, i.e. $a \rightarrow b = 0$ iff $a = 1$ and $b = 0$. Multiplication $\otimes$ and residuum $\rightarrow$ are intended for modeling the conjunction and implication, respectively. Supremum ($\bigvee$) and infimum ($\bigwedge$) are intended for modeling general and existential quantifier, respectively.

An **L**-set (fuzzy set with truth degrees in **L**) [18, 7] A in a universe set X is any map $A : X \to L$. By L^X we denote the set of all **L**-sets in X. The concept of **L**-relation is defined obviously. By $\{\,{}^{a}\!/x\}$ we denote an **L**-set in X such that $\{\,{}^{a}\!/x\}(x) = a$ and $\{\,{}^{a}\!/x\}(y) = 0$ for $y \neq x$. Operations on L extend pointwise to L^X, e.g. $(A \vee B)(x) = A(x) \vee B(x)$ for $A, B \in L^X$. Following common usage, we write $A \cup B$ instead of $A \vee B$, etc. Given $A, B \in L^X$, the subsethood degree $S(A, B)$ of A in B is defined by $S(A, B) = \bigwedge_{x \in X} A(x) \to B(x)$. We write $A \subseteq B$ if $S(A, B) = 1$. Analogously, the equality degree $(A \approx B)$ of A and B is defined by $(A \approx B) = \bigwedge_{x \in X}(A(x) \leftrightarrow B(x))$ where $\leftrightarrow$ is the so-called biresiduum defined by $a \leftrightarrow b = (a \to b) \wedge (b \to a)$. It is immediate that $A \approx B = S(A, B) \wedge S(B, A)$. For $a \in L$ and $A \in L^X$, the ordinary set ${}^{a}A = \{x \in X \mid A(x) \geq a\}$ is called the a-cut of A.

A binary **L**-relation $\approx$ on a set X is called an **L**-equivalence (fuzzy equivalence) if

$$x \approx x = 1 \tag{1}$$

$$x \approx y = y \approx x \tag{2}$$

$$x \approx y \otimes y \approx z \leq x \approx z. \tag{3}$$

An **L**-equivalence is called an **L**-equality if

$$x \approx y = 1 \quad \text{implies} \quad x = y. \tag{4}$$

Sometimes, a fuzzy equivalence is called simply a similarity (or fuzzy similarity). We will use the term similarity (or **L**-similarity) as well. There has been a lot of debates about what properties a relation modeling similarity should have. It is mostly agreed that similarity is reflexive and symmetric. However, transitivity of similarity has been a point of disagreement. One usually argues against transitivity as follows: If similarity were transitive then any two colors would be similar. For we may suppose that two colors with sufficiently close wave lengths are similar. Now, for any two colors A and B we may find a chain $A = A_1, A_2, \ldots, A_n = B$, of colors such that A_i and A_{i+1} are similar. Using transitivity, A and B are similar. On the other hand, the transitivity condition formulated verbally (i.e. "if x and y are similar, and if y and z are similar then x and z are similar") sounds plausible. The solution to this puzzle lies in the fact that similarity, by its nature, is a graded (fuzzy) notion. If we look at the meaning of transitivity in fuzzy setting, we find it quite natural. For example, if $E(x, y) = 0.8$ (x and y are similar in degree 0.8) and $E(y, z) = 0.8$ (y and z are similar in degree 0.8) then x and z have to be similar at least in degree $0.8 \otimes 0.8$. Thus, in case of the product structure, transitivity forces $E(x, z) \geq 0.8 \otimes 0.8 = 0.64$ which is in accord with our intuitive feeling. Note however, that Gödel t-norm does not help (which is due to its idempotence).

The next theorem shows a universal way to construct similarity.

Theorem 2.1. ([16, 2])
A binary **L**-relation $\approx$ on X is an **L**-equivalence iff there is $\mathcal{S} \subseteq L^X$ of **L**-sets in X such that $\approx = \approx_{\mathcal{S}}$ where

$$(x \approx_{\mathcal{S}} y) = \bigwedge_{A \in \mathcal{S}} (A(x) \leftrightarrow A(y)). \tag{5}$$

The elements A of $\mathcal{S}$ represent fuzzy attributes ($A(x)$ is the degree to which an element x has the attribute A). Therefore, Theorem 2.1 says that "x and y are considered similar if and only if for each (relevant, i.e. belonging to $\mathcal{S}$) attribute A we have that x has A iff y has A". This rule is a modification of the well-known Leibniz criterion [1] of identity. A useful corollary of Theorem 2.1 says that $\approx$ (cf. equality degree introduced above) is an **L**-equivalence relation on L^X.

An **L**-set A in X is said to be compatible with an **L**-equivalence $\approx$ on X if $A(x) \otimes (x \approx y) \leq A(y)$ for each $x, y \in X$ (that is, "if x belongs to A and x and y are similar then y belongs to A" is true). The collection of all **L**-sets in X compatible with $\approx$ will be denoted by $L^{\langle X, \approx \rangle}$.

3. Fuzzy closure induced by similarity

Coming to similarity-based fuzzy closure Having introduced the necessary formal notions, we can go back to the motivating examples from Section 1. The set $L^{\langle X, \approx \rangle}$ contains exactly those fuzzy sets in X that respect $\approx$. For a given fuzzy set A in X it might thus be desirable to know the least fuzzy set $\mathrm{C}(A)$ which both contains A and is compatible with $\approx$. Since $L^{\langle X, \approx \rangle}$ is closed under intersections, we have

$$\mathrm{C}(A) = \bigcap \{B \mid B \in L^{\langle X, \approx \rangle},\ A \subseteq B\}.$$

It is easy to see that $\mathrm{C}(A)$ may be described directly using $\approx$ by

$$\mathrm{C}(A)(y) = \bigvee_{x \in X} A(x) \otimes (x \approx y). \tag{6}$$

Without going into details, we note that except the above-mentioned fact that considering only fuzzy sets compatible with $\approx$ is sound from the epistemic point of view (once one interprets $\approx$ as an underlying indistinguishability), there are more "technical" reasons: when $\approx$ is employed, fuzzy sets compatible with $\approx$ behave intuitively well. For example, given any first-order formula φ with free variables $x, \ldots, y$ whose relation symbols are interpreted by fuzzy sets compatible with $\approx$, one can naturally estimate the truth degree to which $\varphi[u, \ldots, v]$ (i.e., variables $x, \ldots, y$ evaluate to elements $u, \ldots, v$ of the universe) is equivalent to $\varphi[u', \ldots, v']$ in terms of similarity degrees $u \approx u'$, …, $v \approx v'$, see [10].

If X is a set of elements of a database then a user-query may be given by listing some examples representing appropriate results for the query, each with a degree to which it is appropriate. That is, the query may be given by a fuzzy set A in X ($A(x)$ is the degree to which x is appropriate). Naturally, we expect the answer to the query A to contain those elements from X which are similar to some example from A. In other words, we expect that the degree $\mathrm{Ans}(A)(y)$ to which an element y from X belongs to the answer $\mathrm{Ans}(A)$ is the truth degree of the fact "there is an element x in A such that x and y are similar". Basic rules of semantics of fuzzy logic tell that $\mathrm{Ans}(A)(y)$ is just equal to $\mathrm{C}(A)(y)$ defined in (6).

From the above examples it is clear that the operator of a similarity-based closure assigning a fuzzy set $\mathrm{C}(A)$ to a fuzzy set A is an important one. Our aim in the following is twofold. First, we discuss some relationships between similarity-based closure and metric closure. Second, we investigate abstract properties of the similarity-based closure operator and provide its complete axiomatization.

Relationship to metric closure The concept of a metric is the one mostly applied when considering closeness of objects (usually of a geometric nature). There is an obvious question of what is the relationship between the notion of a metric and the notion of an **L**-similarity (i.e. a fuzzy equivalence). After we recall known relationships, we turn to the second immediate question of the relationship between the well-established concept of a metric closure and that of a similarity-based closure.

We start with some relationships between metrics and **L**-similarities. Recall that a metric on a nonempty set X is a mapping δ assigning to any $x, y \in X$ a nonnegative real $\delta(x, y)$ such that

$$\delta(x,y) = 0 \text{ iff } x = y,$$
$$\delta(x,y) = \delta(y,x),$$
$$\delta(x,z) \leq \delta(x,y) + \delta(y,z).$$

If instead of the first condition we require only $\delta(x, x) = 0$, we get a more general notion of a pseudometric. Sometimes, (pseudo)metric is used in a generalized sense allowing to assign also value ∞ (infinite distance). In that case we speak of a generalized (pseudo)metric. A (generalized) (pseudo)metric space is a pair $\langle X, \delta\rangle$ where X is a nonempty set and δ a (generalized) (pseudo)metric on X.

Basically, a metric maps to $[0, \infty)$ while a fuzzy similarity maps in general to a support L of a complete residuated lattice. However, if one restricts the consideration to residuated lattices over $[0, 1]$ (i.e. to left-continuous t-norms), interesting relationships come out. Some relationships are illustrated in the following. To have a suitable analogy in terminology, call an *(***L***)-similarity space* a pair $\mathbf{X} = \langle X, \approx\rangle$ where $\approx$ is an **L**-equivalence on X. $\mathbf{X}$ is called *strict* if $\approx$ is an **L**-equality.

Example 3.1. (1) Let **L** be the standard Łukasiewicz algebra on $[0, 1]$ (i.e. $\otimes$ is the Łukasiewicz t-norm), let $\mathbf{X} = \langle X, \approx\rangle$ be a similarity space. Put $\delta_\approx(x, y) = \neg(x \approx y)$. Then $\delta_\approx(x, x) = 1 - (x \approx x) = 0$; $\delta_\approx(x, y) = \delta_\approx(y, x)$; $\delta_\approx(x, y) = \neg(x \approx y) \leq \neg((x \approx z) \otimes (z \approx y)) = \neg(x \approx z) \oplus \neg(z \approx y) = \min(\neg(x \approx z) + \neg(z \approx y), 1) \leq \neg(x \approx z) + \neg(z \approx y) = \delta_\approx(x, y) + \delta_\approx(y, z)$. Thus, $\delta_\approx$ is a pseudometric on X with $\delta_\approx(x, y) \in [0, 1]$. If $\mathbf{X}$ is, moreover, strict then $\delta_\approx$ is a metric.

Conversely, if δ is a (pseudo)metric on X with $\delta(x, y) \in [0, 1]$ then putting $(x \approx_\delta y) = 1 - \delta(x, y)$ we get that $\mathbf{X}_\delta = \langle X, \approx_\delta\rangle$ is a (strict) similarity space (transitivity: $(x \approx_\delta y) \otimes (y \approx_\delta z) = \neg\delta(x, y) \otimes \neg\delta(y, z) = \neg(\delta(x, y) \oplus \delta(y, z)) = \neg \min(\delta(x, y) + \delta(y, z), 1) \leq \neg(\min(\delta(x, z), 1)) = \neg(\delta(x, z)) = (x \approx_\delta z)$).

(2) Let **L** be the standard product algebra on $[0, 1]$ (i.e. $\otimes$ is the product t-norm). For a similarity space $\mathbf{X} = \langle X, \approx\rangle$, ${}^{0+}\!\approx = \{\langle x, y\rangle \mid (x \approx y) > 0\}$ is an equivalence relation on X with equivalence classes, say, X_i ($i \in I$). On each X_i, put $\delta_\approx(x, y) = -\log(x \approx y)$; for $x \in X_i$, $y \in X_j$ ($X_i \neq X_j$) put $\delta(x, y) = \infty$. Then $\delta_\approx$ is a pseudometric on X (in a generalized sense since it may take also ∞ as its values). Indeed: $\delta_\approx(x, x) = -\log(1) = 0$; $\delta_\approx(x, y) = -\log(x \approx y) = -\log(y \approx x) = \delta_\approx(y, x)$; $\delta_\approx(x, z) = -\log(x \approx z) \leq -\log((x \approx y) \cdot (y \approx z)) = (-\log(x \approx y)) + (-\log(y \approx z)) = \delta_\approx(x, y) + \delta_\approx(y, z)$. Moreover, $\delta_\approx$ is a metric iff $\mathbf{X}$ is strict.

The foregoing two examples are special cases of the following general relationship between pseudometrics on X (in the generalized sense) and **L**-equivalences on X where **L** is a residuated lattice on $[0, 1]$ given by a continuous Archimedean t-norm (i.e. $\otimes$ is continuous as a real function and satisfies $a \otimes a < a$ for each $a \neq 0, 1$). We need the following representation theorem for continuous Archimedean t-norms (for proof see e.g. [13]):

Theorem 3.1. A mapping $\otimes : [0, 1]^2 \to [0, 1]$ is a continuous Archimedean t-norm iff there is a continuous additive generator f such that

$$x \otimes y = f^{(-1)}(f(x) + f(y)),$$

i.e. f is a strictly decreasing continuous mapping $f : [0,1] \to [0,\infty]$ with $f(1) = 0$ and $f^{(-1)}$ is the pseudoinverse of f defined by $f^{(-1)}(x) = f^{-1}(x)$ if $x \leq f(0)$ and $f^{(-1)}(x) = 0$ otherwise.

Łukasiewicz as well as product t-norms are both continuous and Archimedean. $f(x) = 1 - x$ and $f^{(-1)}(x) = \max(1-x, 0)$ are an additive generator and its pseudoinverse of the Łukasiewicz t-norm; $f(x) = -\log(x)$ and $f^{(-1)}(x) = e^{-x}$ are an additive generator and its pseudoinverse of the product t-norm. Now, we have the following result which follows by combination of results from [5].

Theorem 3.2. Let $\otimes$ be a continuous Archimedean t-norm with an additive generator f, $\mathbf{L}$ be a residuated lattice on $[0,1]$ given by $\otimes$, $\approx$ be an $\mathbf{L}$-equivalence on X, δ be a pseudometric on X in a generalized sense. Then (1) $\delta_{\approx} : [0,1]^2 \to [0,\infty]$ defined by

$$\delta_{\approx}(x,y) = f(x \approx y)$$

is a pseudometric in a generalized sense which is a metric iff $\approx$ is an $\mathbf{L}$-equality; (2) $\approx_\delta$: $[0,1]^2 \to [0,1]$ defined by

$$(x \approx_\delta y) = f^{(-1)}(\delta(x,y))$$

is an $\mathbf{L}$-equivalence on X which is an $\mathbf{L}$-equality iff δ is a metric; (3) $\approx$ equals $\approx_{\delta_\approx}$ and if $\delta(X,X) \subseteq [0, f(0)]$ then δ equals $\delta_{\approx_\delta}$.

Proof: The result follows by easy combination of results obtained in [5].

(1): $\delta_\approx(x,x) = 0$ and $\delta_\approx(x,y) = \delta_\approx(y,x)$ follow from $(x \approx x) = 1$, $(x \approx y) = (y \approx x)$, and $f(1) = 0$. Triangle inequality for $\delta_\approx$ can be obtained as follows: Transitivity of $\approx$ yields $f^{(-1)}(f(x \approx y) + f(y \approx z)) \leq (x \approx z)$. Since f is decreasing, we have $f(x \approx z) \leq f(f^{(-1)}(f(x \approx y) + f(y \approx z)))$. Now, there are two possibilities: either $f(x \approx y) + f(y \approx z) > f(0)$ and then $f(x \approx z) \leq f(f^{(-1)}(f(x \approx y) + f(y \approx z))) = f(0) < f(x \approx y) + f(y \approx z)$, or $f(x \approx y) + f(y \approx z) \leq f(0)$ and then $f(x \approx z) \leq f(f^{(-1)}(f(x \approx y) + f(y \approx z))) = f(x \approx y) + f(y \approx z)$. In both of the cases we have $f(x \approx z) \leq f(x \approx y) + f(y \approx z)$ which means $\delta_\approx(x,z) \leq \delta_\approx(x,y) + \delta_\approx(y,z)$, i.e. the required triangle inequality.

That $\delta_\approx$ is a metric iff $\approx$ is an $\mathbf{L}$-equality follows easily from the fact that f is strictly decreasing.

(2): $(x \approx_\delta x) = 1$ and $(x \approx_\delta y) = (y \approx_\delta x)$ follow from $\delta(x,x) = 0$, $\delta(x,y) = \delta(y,x)$, and $f^{(-1)}(0) = 1$. Transitivity of $\approx_\delta$: Note first that $f^{(-1)}(x) = f^{-1}(\min(f(0), x))$, $f(u \approx_\delta v) = \min(f(0), \delta(u,v))$, and that f^{-1} is decreasing. Using triangle inequality for δ, we have

$$\begin{aligned} & (x \approx_\delta z) = f^{(-1)}(\delta(x,z)) = f^{-1}(\min(f(0), \delta(x,z))) \geq \\ \geq \quad & f^{-1}(\min(f(0), \delta(x,y) + \delta(y,z))) = \\ = \quad & f^{-1}(\min(f(0), f(x \approx_\delta y) + f(y \approx_\delta z))) = (x \approx y) \otimes (y \approx z), \end{aligned}$$

verifying transitivity of $\approx_\delta$.

Since $f^{(-1)}$ is strictly decreasing, $\approx_\delta$ is an $\mathbf{L}$-equality iff δ is a metric.

(3): $(x \approx_{\delta_\approx} y) = f^{(-1)}(\delta_\approx(x,y)) = f^{(-1)}(f(x \approx y)) = (x \approx y)$.
If $\delta(x,y) \leq f(0)$ then $\delta_{\approx_\delta}(x,y) = f(x \approx_\delta y) = f(f^{(-1)}(\delta(x,y))) = f(f^{-1}(\min(f(0), \delta(x,y)))) = f(f^{-1}(\delta(x,y))) = \delta(x,y)$. □

Next, we discuss some basic relationships between the similarity-based fuzzy closure and metric closure. Recall that if $\mathbf{X} = \langle X, \delta \rangle$ is a (pseudo)metric space and $A \subseteq X$ then the set $\mathrm{C}_\delta(A)$ defined by

$$\mathrm{C}_\delta(A) = \{y \mid \text{ for each } \varepsilon > 0 \text{ there is } x \in A : \delta(x, y) < \varepsilon\}$$

is called the closure of A in $\mathbf{X}$. Our aim is to discuss the relationship between C_δ and $\mathrm{C}_\approx$ where $\approx$ is a fuzzy similarity corresponding (somehow) to δ and $\mathrm{C}_\approx$ is the $\approx$-based operator defined by (6). It is important to realize that C_δ is a mapping from 2^X to 2^X (i.e. operating on ordinary sets) while $\mathrm{C}_\approx$ is a mapping from L^X to L^X (i.e. operating on fuzzy sets). If δ is a (generalized) pseudometric, then for any $a \in (0, \infty]$, the mapping $\delta_a : \langle x, y \rangle \mapsto \min(a, \delta(x, y))$ is a (generalized) pseudometric as well (easy to verify). Moreover, it is easy to show that $\mathrm{C}_\delta = \mathrm{C}_{\delta_a}$. Therefore, if we are interested in the metric closure only, we may restrict our attention to (generalized) (pseudo)metrics with $\delta(X, X) \subseteq [0, f(0)]$ where f is the generator of a given continuous Archimedean t-norm $\otimes$. Namely, $\mathrm{C}_\delta = \mathrm{C}_{\delta_{f(0)}}$ and, due to Theorem 3.2, there is a one-to-one correspondence between (generalized) (pseudo)metrics satisfying $\delta(X, X) \subseteq [0, f(0)]$ and $\mathbf{L}$-similarities where $\mathbf{L}$ is given by the corresponding t-norm $\otimes$. The next theorem shows a way to describe C_δ using $\mathrm{C}_\approx$.

Theorem 3.3. Let $\otimes$ be a continuous Archimedean t-norm with a continuous additive generator f, let δ be a generalized (pseudo)metric and $\approx$ be an $\mathbf{L}$-similarity corresponding to δ, i.e. $\approx = \approx_\delta$ and $\delta = \delta_\approx$ (cf. Theorem 3.2). Then

$$\mathrm{C}_\delta(A) = {}^1(\mathrm{C}_\approx(A))$$

for each $A \subseteq X$. Furthermore,

$$x \in \mathrm{C}_\delta(A) \text{ iff for each } \varepsilon < 1 : A \cap {}^\varepsilon\mathrm{C}_\approx(\{\,{}^1/x\}) \neq \emptyset$$

for each $A \subseteq X$, $x \in X$.

Proof: First, we show $\mathrm{C}_\delta(A) = {}^1(\mathrm{C}_\approx(A))$: We have $y \in {}^1(\mathrm{C}_\approx(A))$ iff $1 = \bigvee_{x \in A}(x \approx y) = \bigvee_{x \in A} f^{(-1)}(\delta(x, y))$. That is, for each $\eta > 0$ there is $x \in A$ such that $f^{(-1)}(\delta(x, y)) > 1 - \eta$ which is equivalent to saying that for each η with $f(0) \geq \eta > 0$ there is $x \in A$ such that $f^{(-1)}(\delta(x, y)) > 1 - \eta$. Since $f^{(-1)} : [0, f(0)] \to [0, 1]$, is the inverse function to $f : [0, 1] \to [0, f(0)]$, the latter condition is equivalent to saying that for each η with $f(0) \geq \eta > 0$ there is $x \in A$ such that $\delta(x, y) < f(1 - \eta)$. Now, since for $\varepsilon = f(1 - \eta)$ we have that $\varepsilon \to 0$ iff $f(1 - \eta) \to 0$, we further have that the latter condition holds iff for each $\varepsilon > 0$ there is $x \in A$ such that $\delta(x, y) < \varepsilon$ which is equivalent to $y \in \mathrm{C}_\delta(A)$.

Next, $A \cap {}^\varepsilon\mathrm{C}_\approx(\{\,{}^1/x\}) \neq \emptyset$ for each $\varepsilon < 1$ holds iff for each $\varepsilon < 1$ there is $y \in A$ such that $\varepsilon \leq (x \approx y)$ which means that $\varepsilon \leq f^{(-1)}(\delta(x, y))$. For ε sufficiently close to 1, $\varepsilon \leq f^{(-1)}(\delta(x, y))$ is equivalent to $\delta(x, y) \leq f(\varepsilon)$. Since we have $\varepsilon \to 1$ (from left) iff $f(\varepsilon) \to 0$ (from right), saying that for each $\varepsilon < 1$ there is $y \in A$ such that $\delta(x, y) \leq f(\varepsilon)$ is equivalent to saying that for each $\eta = f(\varepsilon) > 0$ there is $y \in A$ such that $\delta(x, y) \leq \eta$ which means that $x \in \mathrm{C}_\delta(A)$. To sum up, $x \in \mathrm{C}_\delta(A)$ iff for each $\varepsilon < 1$: $A \cap {}^\varepsilon\mathrm{C}_\approx(\{\,{}^1/x\}) \neq \emptyset$. □

Axiomatic characterization of similarity-based closure operators The fuzzy set $\mathrm{C}(A)$ is often called the extensional hull (or closure) of A. In what follows we consider the operator C from the point of view of closure operators of fuzzy sets as studied in [3, 4], see also [6]. The following is a useful definition.

Definition 3.1. Let $\mathbf{X} = \langle X, \approx\rangle$ be an $\mathbf{L}$-similarity space. For $\mathbf{L}$-sets $A, B \in L^X$, we put

$$\rho_{\mathbf{X}}(A,B) = \bigvee_{x,y\in X} (A(x) \otimes (x \approx y) \otimes B(y)).$$

Remark 3.1. (1) $\rho_{\mathbf{X}}(A,B)$ is naturally interpreted as the truth degree of the fact that there are some x in A and y in B which are similar.

(2) One can easily see that $\rho_{\mathbf{X}}$ extends $\approx$ in that $(x \approx y) = \rho_{\mathbf{X}}(\{\,{}^1/x\}, \{\,{}^1/y\})$.

(3) $\rho_{\mathbf{X}}$ is a symmetric relation on L^X which is not transitive in general. It is easy to see that if $A \in L^{\langle X,\approx\rangle}$ or $B \in L^{\langle X,\approx\rangle}$ then $\rho_{\mathbf{X}}(A,B) = \bigvee_{x\in X}(A(x) \otimes B(x))$, i.e. $\rho_{\mathbf{X}}(A,B)$ is the height of $A \otimes B$.

An immediate verification shows that introducing $\mathrm{C}_{\mathbf{X}} : L^X \to L^X$ by

$$\mathrm{C}_{\mathbf{X}}(A)(x) = \rho_{\mathbf{X}}(\{\,{}^1/x\}, A). \tag{7}$$

we have $\mathrm{C}_{\mathbf{X}}(A)(x) = \bigvee_{y\in X} A(y) \otimes (x \approx y)$ which is the definition (6) of the similarity-based closure of A. We will freely use any of $\mathrm{C}_{\mathbf{X}}$, $\mathrm{C}_{\approx}$, and C to denote the operator we are dealing with.

Recall the following definition and basic results from [3].

Definition 3.2. Let $K \subseteq L$ be a $\leq$-filter (i.e. $K \neq \emptyset$, and $a \in K$, $a \leq b$ imply $b \in K$). An $\mathbf{L}_K$-*closure operator* on a set X is a mapping $\mathrm{C} : L^X \to L^X$ satisfying

$$A \subseteq \mathrm{C}(A) \tag{8}$$
$$S(A_1, A_2) \leq S(\mathrm{C}(A_1), \mathrm{C}(A_2)) \quad \text{whenever } S(A_1, A_2) \in K \tag{9}$$
$$\mathrm{C}(A) = \mathrm{C}(\mathrm{C}(A)) \tag{10}$$

for every $A, A_1, A_2 \in L^X$.

Remark 3.2. (1) Definition 3.2 generalizes some earlier approaches to fuzzy closure operators [6], mainly in that it takes into account partial subsethood (sensitivity to partial subsethood is parametrized by K). Particularly, for $L = [0,1]$, $\mathbf{L}_{\{1\}}$-closure operators are precisely fuzzy closure operators [6]. If $K = L$, we omit the subscript K and use the term $\mathbf{L}$-closure operator.

(2) It is easily seen that for $\mathbf{L} = \mathbf{2}$ (classical logic), the notion of an $\mathbf{L}_K$-closure operator coincides with the notion of a closure operator.

The next theorem gives a characterization of a system of closed fuzzy sets of $\mathbf{L}$-closure operators (see [3]). For $\mathrm{C} : L^X \to L^X$ denote $\mathcal{S}_{\mathrm{C}} = \{A \mid A = \mathrm{C}(A)\}$.

Theorem 3.4. $\mathcal{S}$ is a system of all closed fuzzy sets of some $\mathbf{L}$-closure operator C, i.e. $\mathcal{S} = \mathcal{S}_{\mathrm{C}}$, iff $\mathcal{S}$ is closed under arbitrary intersections and a-shifts, i.e. for $A_i, A \in \mathcal{S}$, $a \in L$, we have $\cap_i A_i \in \mathcal{S}$ and $a \to A \in \mathcal{S}$.

Remark 3.3. A system $\mathcal{S}$ of $\mathbf{L}$-sets in X is called an $\mathbf{L}$-closure system in X if it is closed under arbitrary intersections and a-shifts. In [3] it is shown that $\mathrm{C} \mapsto \mathcal{S}_{\mathrm{C}}$ and $\mathcal{S} \mapsto \mathrm{C}_{\mathcal{S}}$, where $\mathrm{C}_{\mathcal{S}}(A) = \bigcap\{B \in \mathcal{S} \mid A \subseteq B\}$, establish a bijective correspondence between $\mathbf{L}$-closure operators and $\mathbf{L}$-closure systems in X. Moreover, for an $\mathbf{L}$-closure system $\mathcal{S}$ we have

$$\bigcap\{B \in \mathcal{S} \mid A \subseteq B\} = \bigcap_{B\in\mathcal{S}} S(A,B) \to B,$$

see [3].

We now proceed to show that similarity-based closures are exactly $\mathbf{L}$-closure operators satisfying three additional properties.

Lemma 3.1. Let $\mathbf{X} = \langle X, \approx\rangle$ be an $\mathbf{L}$-similarity space. Then the mapping $\mathrm{C}_{\mathbf{X}}$ defined by (7) is an $\mathbf{L}$-closure operator satisfying, moreover,

$$\mathrm{C}_{\mathbf{X}}(\bigcup_{i\in I} A_i) = \bigcup_{i\in I} \mathrm{C}_{\mathbf{X}}(A_i), \tag{11}$$

$$\mathrm{C}_{\mathbf{X}}(\{\,{}^{a}/x\}) = a \otimes \mathrm{C}_{\mathbf{X}}(\{\,{}^{1}/x\}), \tag{12}$$

$$\mathrm{C}_{\mathbf{X}}(\{\,{}^{1}/x\})(y) = \mathrm{C}_{\mathbf{X}}(\{\,{}^{1}/y\})(x) \tag{13}$$

for any $A_i \in L^X$ ($i \in I$), $x, y \in X$, $a \in L$.

Proof:
We have $A(x) = A(x) \otimes (x \approx x) \leq \bigvee_{y\in X} A(x) \otimes (x \approx y) = \rho_{\mathbf{X}}(\{\,{}^{1}/x\}, A)$, thus $A \subseteq \mathrm{C}_{\mathbf{X}}(A)$, proving (8).

(9) is true iff for each $x \in X$ and every $A, B \in L^X$ we have $S(A,B) \leq \mathrm{C}_{\mathbf{X}}(A)(x) \to \mathrm{C}_{\mathbf{X}}(B)(x)$ which is equivalent to $\mathrm{C}_{\mathbf{X}}(A)(x) \otimes S(A,B) \leq \mathrm{C}_{\mathbf{X}}(B)(x)$. The last inequality is true. Indeed,

$$\begin{aligned}
\mathrm{C}_{\mathbf{X}}(A)(x) \otimes S(A,B) &= (\bigvee_{y\in X} A(y) \otimes (x \approx y)) \otimes (\bigwedge_{y\in X} A(y) \to B(y)) \leq \\
&\leq \bigvee_{y\in X} (A(y) \otimes (A(y) \to B(y)) \otimes (x \approx y)) \leq \\
&\leq \bigvee_{y\in X} (B(y) \otimes (x \approx y)) = \mathrm{C}_{\mathbf{X}}(B)(x)
\end{aligned}$$

proving (9).

In order to show (10), we proceed as follows:

$$\begin{aligned}
(\mathrm{C}_{\mathbf{X}}(\mathrm{C}_{\mathbf{X}}(A)))(x) &= \bigvee_{z\in X} \mathrm{C}_{\mathbf{X}}(A)(z) \otimes (x \approx z) = \bigvee_{z\in X} (\bigvee_{y\in X} A(y) \otimes (z \approx y)) \otimes (x \approx z) = \\
&= \bigvee_{z\in X} \bigvee_{y\in X} (A(y) \otimes (x \approx z) \otimes (z \approx y)) \leq \\
&\leq \bigvee_{z\in X} \bigvee_{y\in X} (A(y) \otimes (x \approx y)) = \bigvee_{y\in X} (A(y) \otimes (x \approx y)) = \mathrm{C}_{\mathbf{X}}(A)(x)
\end{aligned}$$

proving (10).

Furthermore, we have

$$\mathrm{C}_{\mathbf{X}}(\bigcup_{i\in I} A_i) = \bigvee_{y\in Y}((\bigvee_{i\in I} A_i(y))\otimes(x\approx y)) = \bigvee_{i\in I}(\bigvee_{y\in Y} A_i(y)\otimes(x\approx y)) = (\bigcup_{i\in I}\mathrm{C}_{\mathbf{X}}(A_i))(x)$$

verifying (11).

(12) is true since

$$\mathrm{C}_{\mathbf{X}}(\{\,{}^{a}/x\})(y) = \bigvee_{z\in X}(\{\,{}^{a}/x\}(z)\otimes(z\approx y)) = a\otimes(x\approx y) = a\otimes\mathrm{C}_{\mathbf{X}}(\{\,{}^{1}/x\})(y).$$

Finally, (13) follows directly from symmetry of $\approx$. □

Remark 3.4. Note that for any operator $\mathrm{C} : L^X \to L^X$, (12) implies $\mathrm{C}(\emptyset) = \emptyset$. Indeed, $\mathrm{C}(\emptyset)(x) = \mathrm{C}(\{\,{}^{0}/y\}) = 0\otimes\mathrm{C}(\{\,{}^{1}/y\}) = 0$ for any $x, y \in X$.

Remark 3.5. In the following, we repeatedly use a simple fact that $A = \bigcup_{x\in X}\{\,{}^{A(x)}/x\}$ for each $A \in L^X$.

Lemma 3.2. Let C be an **L**-closure operator on X that satisfies (11)–(13). For $x, y \in X$ put

$$(x \approx_{\mathrm{C}} y) = \mathrm{C}(\{\,{}^{1}/x\})(y).$$

Then $\mathbf{X}_{\mathrm{C}} = \langle X, \approx_{\mathrm{C}}\rangle$ is an **L**-similarity space.

Proof: We verify (1): $(x \approx_{\mathrm{C}} x) = \mathrm{C}(\{\,{}^{1}/x\})(x) \geq \{\,{}^{1}/x\}(x) = 1$, by (8).

By (13), $(x \approx_{\mathrm{C}} y) = \mathrm{C}(\{\,{}^{1}/x\})(y) = \mathrm{C}(\{\,{}^{1}/y\})(x) = (y \approx_{\mathrm{C}} x)$ proving (2).

Take $x, z \in X$ and put $A = \{\,{}^{1}/x\}$. We have $\mathrm{C}(A)(z) = (x \approx_{\mathrm{C}} z)$ and by (11) and (12),

$$\begin{aligned}\mathrm{C}(\mathrm{C}(A))(z) &= \mathrm{C}(\bigcup_{y\in X}\{\,{}^{\mathrm{C}(\{\,{}^{1}/x\})(y)}/y\})(z) = \bigvee_{y\in X}\mathrm{C}(\{\,{}^{\mathrm{C}(\{\,{}^{1}/x\})(y)}/y\})(z) = \\ &= \bigvee_{y\in X}\mathrm{C}(\{\,{}^{1}/x\})(y)\otimes\mathrm{C}(\{\,{}^{1}/y\})(z) = \bigvee_{y\in X}(x\approx_{\mathrm{C}} y)\otimes(y\approx_{\mathrm{C}} z).\end{aligned}$$

On account of (10) we have $\mathrm{C}(\mathrm{C}(A))(z) \leq \mathrm{C}(A)(z)$, i.e.

$$\bigvee_{y\in X}(x\approx_{\mathrm{C}} y)\otimes(y\approx_{\mathrm{C}} z) \leq (x\approx_{\mathrm{C}} z).$$

From this it follows that $(x\approx_{\mathrm{C}} y)\otimes(y\approx_{\mathrm{C}} z) \leq (x\approx_{\mathrm{C}} z)$ for any $y \in X$, establishing (3). □

Theorem 3.5. The mappings sending $\mathbf{X}$ to $\mathrm{C}_{\mathbf{X}}$, and C to $\mathbf{X}_{\mathrm{C}}$, as defined in Lemmas 3.1 and 3.2, are mutually inverse mappings between the set of all **L**-similarity spaces with support X and the set of all **L**-closure operators on X satisfying (11)–(13).

Proof: By Lemmas 3.1 and 3.2, we have to check that $\mathbf{X} = \mathbf{X}_{\mathrm{C}_{\mathbf{X}}}$ and $\mathrm{C} = \mathrm{C}_{\mathbf{X}_{\mathrm{C}}}$. We have

$$(x \approx_{\mathrm{C}_{\mathbf{X}}} y) = \mathrm{C}_{\mathbf{X}}(\{\,{}^{1}/x\})(y) = \rho_{\mathbf{X}}(\{\,{}^{1}/y\},\{\,{}^{1}/x\}) = (x \approx y).$$

Furthermore,

$$\begin{aligned}\mathrm{C}(A)(x) &= \mathrm{C}(\bigcup_{y\in X}\{\,{}^{A(y)}/y\})(x) = \bigvee_{y\in X}\mathrm{C}(\{\,{}^{A(y)}/y\})(x) = \\ &= \bigvee_{y\in X} A(y)\otimes \mathrm{C}(\{\,{}^{1}/y\})(x) = \bigvee_{y\in X} A(y)\otimes(x\approx_{\mathrm{C}} y) = \mathrm{C}_{\mathbf{X}_{\mathrm{C}}}(A)(x)\end{aligned}$$

completing the proof. □

As mentioned in Remark 3.3, there is a bijective correspondence between fuzzy closure operators and fuzzy closure systems. Fuzzy closure systems in X are easily axiomatized, see Remark 3.3. In the following we find a suitable axiomatization of systems of closed elements of fuzzy closure operators which are induced by similarity spaces.

Lemma 3.3. $A = \mathrm{C}_{\approx}(A)$ iff A is compatible with $\approx$. Thus, $\mathcal{S}_{\mathrm{C}_{\approx}} = L^{\langle X,\approx\rangle}$.

Proof: The statement is a consequence of (6).

Let $A = \mathrm{C}_{\approx}(A)$. Then $A(x)\otimes(x\approx y) = \mathrm{C}_{\approx}(A)(x)\otimes(x\approx y) = \bigvee_{x'}(A(x')\otimes(x'\approx x))\otimes(x\approx y) = \bigvee_{x'}(A(x')\otimes(x'\approx x)\otimes(x\approx y)) \le \bigvee_{x'}(A(x')\otimes(x'\approx y)) = \mathrm{C}_{\approx}(A)(y) = A(y)$.

Conversely, let A be compatible with $\approx$. Then $\mathrm{C}_{\approx}(A)(x) = \bigvee_{x'}(A(x')\otimes(x'\approx x)) \le \bigvee_{x'} A(x) = A(x)$. □

Theorem 3.6. A system $\mathcal{S}$ of $\mathbf{L}$-sets in X is the system of closed sets of some similarity space (i.e. $\mathcal{S} = L^{\langle X,\approx\rangle}$ for some $\langle X,\approx\rangle$) iff it is an $\mathbf{L}$-closure system satisfying $\bigcup_{i\in I} A_i \in \mathcal{S}$, $a\otimes A \in \mathcal{S}$, and $A\to a \in \mathcal{S}$ for each $A_i, A \in \mathcal{S}$, $a\in L$.

Proof: Due to Theorem 3.4 and Remark 3.3, we have to show that for an $\mathbf{L}$-closure system $\mathcal{S}$, $\mathrm{C}_{\mathcal{S}}$ satisfies (11)–(13) iff (a) $\bigcup_{i\in I} A_i \in \mathcal{S}$, (b) $a\otimes A\in\mathcal{S}$, and (c) $A\to a\in\mathcal{S}$ for each $A_i, A\in\mathcal{S}$, $a\in L$. For simplicity, we write only C instead of $\mathrm{C}_{\mathcal{S}}$. We show the following claims.

(i) (11) is equivalent to (a): Assume (11); then for $A_i\in\mathcal{S}$ we have $\bigcup_i A_i = \bigcup_i \mathrm{C}(A_i) = \mathrm{C}(\bigcup_i A_i) \in \mathcal{S}$. Conversely, if $\bigcup_{i\in I} A_i \in \mathcal{S}$ for $A_i\in\mathcal{S}$, then from $\mathrm{C}(A_i)\in\mathcal{S}$ we get $\bigcup_i \mathrm{C}(A_i)\in\mathcal{S}$ and thus $\bigcup_i \mathrm{C}(A_i) = \mathrm{C}(\bigcup_i \mathrm{C}(A_i)) \supseteq \mathrm{C}(\bigcup_i A_i)$. Since we always have $\bigcup_i \mathrm{C}(A_i) \subseteq \mathrm{C}(\bigcup_i A_i)$, (11) follows.

(ii) (11) and (12) imply (b): Assume (11) and (12). Then for $a\in L$ and $A\in\mathcal{S}$ we have

$$\begin{aligned} a\otimes A &= a\otimes\mathrm{C}(A) = a\otimes\mathrm{C}(\bigcup_{x\in X}\{\,{}^{A(x)}/x\}) = a\otimes\bigcup_{x\in X}\mathrm{C}(\{\,{}^{A(x)}/x\}) = \\ &= a\otimes\bigcup_{x\in X} A(x)\otimes\mathrm{C}(\{\,{}^{1}/x\}) = \bigcup_{x\in X} a\otimes A(x)\otimes\mathrm{C}(\{\,{}^{1}/x\}) = \\ &= \cdots = \bigcup_{x\in X}\mathrm{C}(\{\,{}^{a\otimes A(x)}/x\}) = \mathrm{C}(\bigcup_{x\in X}\{\,{}^{a\otimes A(x)}/x\}) = \mathrm{C}(a\otimes A).\end{aligned}$$

(iii) (b) implies (12): Assume that for $a \in L$ and $A \in \mathcal{S}$ we have $a \otimes A \in \mathcal{S}$. For each **L**-closure system we have $a = S(\{ {}^{1}/x\}, \{ {}^{a}/x\}) \leq S(\mathrm{C}(\{ {}^{1}/x\}), \mathrm{C}(\{ {}^{a}/x\}))$ and thus $a \otimes \mathrm{C}(\{ {}^{1}/x\}) \subseteq \mathrm{C}(\{ {}^{a}/x\})$. On the other hand, $\{ {}^{a}/x\} \subseteq a \otimes \mathrm{C}(\{ {}^{1}/x\})$ and thus $\mathrm{C}(\{ {}^{a}/x\}) \subseteq \mathrm{C}(a \otimes \mathrm{C}(\{ {}^{1}/x\})) = a \otimes \mathrm{C}(\{ {}^{1}/x\})$ (as $a \otimes \mathrm{C}(\{ {}^{1}/x\})$ is C-closed) establishing (12).

(iv) (11)–(13) imply (c): If C satisfies (11)–(13) then by Theorem 3.5, C is induced by some similarity $\approx$ on X and thus, by Lemma 3.3, $A \in \mathcal{S}$ means that A is compatible with $\approx$. In order to show $A \to a \in \mathcal{S}$ we thus need to show that $A \to a$ is compatible with $\approx$. We have $(A \to a)(x) \otimes (x \approx y) \leq (A \to a)(y)$ iff $A(y) \otimes (y \approx x) \otimes (A(x) \to a) \leq a$ which is true.

(v) (c) implies (13): Let $A \to a \in \mathcal{S}$ for $A \in \mathcal{S}$, $a \in L$. We have

$$\begin{aligned} \mathrm{C}(\{ {}^{1}/x\})(y) &= \bigwedge_{A \in \mathcal{S}} S(\{ {}^{1}/x\}, A) \to A(y) = \bigwedge_{A \in \mathcal{S}} A(x) \to A(y) \leq \\ &\leq \bigwedge_{A \in \mathcal{S}, a \in L} (A(x) \to a) \to (A(y) \to a) \leq (\mathrm{C}(\{ {}^{1}/y\})(x) \to \mathrm{C}(\{ {}^{1}/y\})(x)) \to \\ &\to (\mathrm{C}(\{ {}^{1}/y\})(y) \to \mathrm{C}(\{ {}^{1}/y\})(x)) = 1 \to (1 \to \mathrm{C}(\{ {}^{1}/y\})(x)) = \mathrm{C}(\{ {}^{1}/y\})(x) \end{aligned}$$

(we put $A = \mathrm{C}(\{ {}^{1}/y\})$ and $a = \mathrm{C}(\{ {}^{1}/y\})(x)$). Symmetrically, $\mathrm{C}(\{ {}^{1}/y\})(x) \leq \mathrm{C}(\{ {}^{1}/x\})(y)$ establishing (13).

Now, from (11)–(13) we get (a) (by (i)), (b) (by (ii)), and (c) (by (iv)). Conversely, from (a)–(c) we get (11) (by (i)), (12) (by (iii)), and (13) (by (v)). The proof is complete. □

Note that Theorem 3.6 is an analogy of theorems on metrizability of topological spaces (i.e. criteria saying when a topological space is induced by a metric).

Corollary 3.1. A system $\mathcal{S}$ of **L**-sets in X is the system of all **L**-sets in X compatible with some **L**-equivalence $\approx$ on X (i.e. $\mathcal{S} = L^{\langle X, \approx \rangle}$) iff $\mathcal{S}$ satisfies

$\bigcap_{i \in I} A_i \in \mathcal{S}$, $\bigcup_{i \in I} A_i \in \mathcal{S}$

$a \otimes A \in \mathcal{S}$, $a \to A \in \mathcal{S}$, $A \to a \in \mathcal{S}$

for each $A_i, A \in \mathcal{S}$ and $a \in L$.

Proof: Directly by Theorem 3.4 and Theorem 3.6. □

Note that using other methods, the result from Corollary 3.1 is obtained in [12]. Our arguments place this result into the appropriate context of closure operators. If $\mathcal{S}$ satisfies the five conditions of Corollary 3.1, i.e. $\mathcal{S} = L^{\langle X, \approx \rangle}$, then $\approx$ may be obtained from $\mathcal{S}$ by Leibniz rule (5):

Corollary 3.2. If $\mathcal{S}$ satisfies the five conditions of Corollary 3.1 then $\mathcal{S} = L^{\langle X, \approx_{\mathcal{S}} \rangle}$.

Proof:
By Corollary 3.1, $\mathcal{S} = L^{\langle X, \approx \rangle}$ for some **L**-equivalence $\approx$. Since for $[x]_{\approx} \in L^X$ defined by $[x]_{\approx}(y) = x \approx y$ we have $[x]_{\approx} \in \mathcal{S}$, it follows that $(x \approx_{\mathcal{S}} y) \leq [x]_{\approx}(x) \leftrightarrow [x]_{\approx}(y) = (x \approx y)$. On the other hand, for each $A \in \mathcal{S}$ we clearly have $(x \approx y) \leq A(x) \leftrightarrow A(y)$ (by compatibility of A) whence $(x \approx y) \leq (x \approx_{\mathcal{S}} y)$. □

Corollary 3.3. $\langle L^{\langle X,\approx\rangle}, \subseteq\rangle$ is a complete sublattice of $\langle L^X, \subseteq\rangle$.

Acknowledgement Supported by grant no. 201/02/P076 of the GA ČR and partially by grant no. B1137301 of the GA AV ČR.

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