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Fuzzy equational logic

RADIM BĚLOHLÁVEK
Institute for Research and Applications of Fuzzy Modeling, University of Ostrava, Bráfova 7, 701 03 Ostrava, Czech Republic, radim.belohlavek@osu.cz

Abstract. Presented is a completeness theorem for fuzzy equational logic with truth values in a complete residuated lattice: Given a fuzzy set Σ of identities and an identity $p \approx q$, the degree to which $p \approx q$ syntactically follows (is provable) from Σ equals the degree to which $p \approx q$ semantically follows from Σ. Pavelka style generalization of well-known Birkhoff's theorem is therefore established.

Key words: fuzzy logic, fuzzy equality, equational logic

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1 Introduction

Fuzzy logic "in narrow sense" (i.e. logical calculi aiming at modern logic-style formalization of graded approach to truth) has been substantially developed recently. Let us briefly recall: In the early 1960's Zadeh introduced fuzzy sets as a mean for modeling of vagueness present in human description of systems. Goguen wrote two papers [3, 4]; among others he proposed to use complete residuated lattices as the structures of truth values. In his seminal work [10], Pavelka developed Goguen's ideas and investigated graded truth approach to logical as well as metalogical notions; he developed a complete (completeness in grades: degree of provability equals degree of validity) propositional fuzzy logic (later extended to first-order case by Novák [8]). Höhle (see e.g. [7]) developed first-order logic complete w.r.t. semantics defined over complete residuated lattices (completeness of provability w.r.t. 1-truth). Finally, Hájek [5] introduced so-called basic logic (algebraization of which leads to BL-algebras which are special residuated lattices) and three of its most important extensions (Łukasiewicz, Gödel, and product), investigated various logical problems in fuzzy setting, and systematized earlier approaches.

Not much attention has been paid to functional symbols and identities (functional symbols are used by Höhle [7] and Novák et al. [9]; Hájek's recent [6] investigates function symbols in basic logic). The aim of this paper is to present fuzzy equational logic with semantics defined over an arbitrary complete residuated lattice and to show its completeness theorem. Our result generalizes the well-known Birkhoff's result [2] in Pavelka style approach.

In Section 2 we present definitions and the completeness theorem. Section 3 contains the proof and some remarks.

2 Fuzzy equational logic

Take any *complete residuated lattice* $\mathbf{L}$ as the structure of truth values. Recall that a complete residuated lattice is an algebra $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ such that $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice with the least element 0 and the greatest element 1, $\langle L, \otimes, 1\rangle$ is a commutative monoid, and $\otimes, \rightarrow$ form an adjoint pair, i.e. $a \otimes b \leq c$ iff $a \leq b \rightarrow c$ is valid for any $a, b, c \in L$. In what follows, $\mathbf{L}$ always refers to a complete residuated lattice. All properties of complete residuated lattices used in what follows are well known and can be found e.g. in [4]. An $\mathbf{L}$-*set* (or fuzzy set with truth degrees in $\mathbf{L}$) in a universe set U is any mapping $A : U \rightarrow L$, $A(u) \in L$ being interpreted as the truth value of "u belongs to A". For $A_1, A_2 : U \rightarrow L$ we put $A_1 \subseteq A_2$ iff $A_1(u) \leq A_2(u)$ for each $u \in U$. If $U = U_1 \times \cdots \times U_n$, A is called an n-ary $\mathbf{L}$-relation between $U_1, \ldots, U_n$. Recall that $\mathbf{L}$-*equivalence* (or $\mathbf{L}$-similarity) on a set U is a binary $\mathbf{L}$-relation E on U satisfying $E(u, u) = 1$ (reflexivity), $E(u, v) = E(v, u)$ (symmetry), and $E(u, v) \otimes E(v, w) \leq E(u, w)$ (transitivity). An $\mathbf{L}$-equivalence on U for which $E(u, v) = 1$ implies $u = v$ will be called an $\mathbf{L}$-*equality.* A function $f : U^n \rightarrow U$ is said to be *compatible* w.r.t. a binary $\mathbf{L}$-relation R on U if for any $u_1, v_1, \ldots, u_n, v_n \in X$ we have

$$R(u_1, v_1) \otimes \cdots \otimes R(u_n, v_n) \leq R(f(u_1, \ldots, u_n), f(v_1, \ldots, v_n)).$$

A collection F of functional symbols, each with its arity will be called a *type.* Given a complete residuated lattice $\mathbf{L}$, the *language* of $\mathbf{L}$-equational logic consists of a countable set X of variables, a type F, a binary predicate symbol $\approx$ standing for (fuzzy) equality, and a set $\{\overline{a} \mid a \in L\}$ of symbols of truth values (however, since there is no danger of misunderstanding, we identify $\overline{a}$ with a). The set $T(X)$ of all terms over F and a non-empty set X of variables is defined as usual (note that $\overline{a}$ is a term for each $a \in L$). If all the variables occurring in a term p are among $x_1, \ldots, x_n$, we write also $p(x_1, \ldots, x_n)$. An *identity* is a formula $p \approx q$ where $p, q \in T(X)$.

Identities will be interpreted in $\mathbf{L}$-algebras: An *algebra of type F with* $\mathbf{L}$-*equality* (or simply an $\mathbf{L}$-algebra of type F) is a triple $\mathbf{A} = \langle A, \approx^{\mathbf{A}}, F^{\mathbf{A}}\rangle$ such that $\langle A, F^{\mathbf{A}}\rangle$ is an algebra of type F (i.e. $F^{\mathbf{A}} = \{f^{\mathbf{A}} : A^n \rightarrow A \mid f \in F \text{ is } n\text{-ary}\}$) and $\approx^{\mathbf{A}}$ is an $\mathbf{L}$-equality on A such that each $f^{\mathbf{A}} \in F^{\mathbf{A}}$ is compatible w.r.t. $\approx^{\mathbf{A}}$. Let $\mathbf{A}$ be an $\mathbf{L}$-algebra, $v : X \rightarrow A$ be a valuation. The interpretation of terms is defined as usual (we denote $\| p \|_{\mathbf{A},v}$ the element of A assigned to the term p by the interpretation). The degree $\| p \approx q \|_{\mathbf{A},v}$ to which $p \approx q$ is true in $\mathbf{A}$ under v is defined by

$$\| p \approx q \|_{\mathbf{A},v} \quad = \quad \| p \|_{\mathbf{A},v} \approx^{\mathbf{A}} \| q \|_{\mathbf{A},v} .$$

The degree $\| p \approx q \|_{\mathbf{A}}$ to which $p \approx q$ is true in $\mathbf{A}$ is defined by

$$\| p \approx q \|_{\mathbf{A}} \quad = \quad \bigwedge_{v:X \rightarrow A} \| p \approx q \|_{\mathbf{A},v},$$

and more generally, if $\mathcal{K}$ is a class of **L**-algebras of type F, we put

$$\| p \approx q \|_{\mathcal{K}} \quad = \quad \bigwedge_{\mathbf{A} \in \mathcal{K}} \| p \approx q \|_{\mathbf{A}} .$$

Given an **L**-set Σ of identities and an identity $p \approx q$, we define the *degree* $\Sigma \models p \approx q$ *of semantical consequence* of $p \approx q$ of Σ by

$$\Sigma \models p \approx q \quad = \quad \bigwedge_{\mathbf{A}:(\forall r,s \in T(X))(\Sigma(r \approx s) \leq \| r \approx s \|_{\mathbf{A}})} \| p \approx q \|_{\mathbf{A}},$$

i.e. the infimum is taken over all **L**-algebras **A** of type F for which any identity $r \approx s$ is true at least in degree $\Sigma(r \approx s)$.

We are to define the notion of a syntactical consequence. A pair $\langle p \approx q, a \rangle$ where $p, q \in T(X)$ and $a \in L$ will be called an (**L**-)*evaluated identity.* Inference rules are defined on evaluated identities as follows (p, q, r, s refer to terms, x is a variable, a, b are from L; $t(u/w)$ denotes the term resulting from a term t by substitution of w for every occurrence of u):

(RE) (from the empty set of evaluated identities) infer $\langle p \approx p, 1 \rangle$ (reflexivity);

(SY) from $\langle p \approx q, a \rangle$ infer $\langle q \approx p, a \rangle$ (symmetry);

(TR) from $\langle p \approx q, a \rangle$ and $\langle q \approx r, b \rangle$ infer $\langle p \approx r, a \otimes b \rangle$ (transitivity);

(REP) from $\langle p \approx q, a \rangle$ infer $\langle r \approx s, a \rangle$ where r is a term containing p as a subterm and s results from r by replacing one occurence of p by q (replacement);

(SUB) from $\langle p \approx q, a \rangle$ infer $\langle p(x/r) \approx q(x/r), a \rangle$ (substitution).

An (**L**-)*evaluated proof* of $p \approx q$ from an **L**-set Σ of identities is any sequence $\langle p_1 \approx q_1, a_1 \rangle, \ldots, \langle p_n \approx q_n, a_n \rangle$ of evaluated identities such that $p_n = p$, $q_n = q$, and for each $i = 1, \ldots, n$, either $a_i = \Sigma(p_i \approx q_i)$ or $\langle p_i \approx q_i, a_i \rangle$ follows from some $\langle p_j \approx q_j, a_j \rangle$('s), $j < i$, by some of the inference rules (RE)–(SUB). In such a case, a_n is called the *value* of the proof. The *degree* $\Sigma \vdash p \approx q$ *of provability* of an identity $p \approx q$ from an **L**-set Σ of identities is defined by

$$\begin{aligned} \Sigma \vdash p \approx q \quad = \quad & \bigvee \{ a_n \mid \langle p_1 \approx q_1, a_1 \rangle, \ldots, \langle p_n \approx q_n, a_n \rangle \text{ is} \\ & \qquad \text{an evaluated proof of } p \approx q \text{ from } \Sigma \}. \end{aligned}$$

The balance of the degree of semantical consequence and the degree of provability is the subject of the following assertion.

Theorem 1 (completeness of L-equational logic) *For any* **L***-set* Σ *of identities and any* $p, q \in T(X)$ *we have* $\Sigma \models p \approx q \; = \; \Sigma \vdash p \approx q$.

The proof will be elaborated in the next section.

3 Proof and some remarks

A homomorphism from an $\mathbf{L}$-algebra $\mathbf{A} = \langle A, \approx^{\mathbf{A}}, F^{\mathbf{A}} \rangle$ into an $\mathbf{L}$-algebra $\mathbf{B} = \langle B, \approx^{\mathbf{B}}, F^{\mathbf{B}} \rangle$ is any mapping $h : A \to B$ satisfying $x \approx^{\mathbf{A}} y \leq h(x) \approx^{\mathbf{B}} h(y)$ (compatibility w.r.t. equality) and $h(f^{\mathbf{A}}(x_1, \ldots, x_n)) = f^{\mathbf{B}}(h(x_1), \ldots, h(x_n))$ (compatibility w.r.t. operations).

If $\mathbf{A} = \langle A, \approx^{\mathbf{A}}, F^{\mathbf{A}} \rangle$ is an $\mathbf{L}$-algebra, an $\mathbf{L}$-equivalence θ on A is called an $\mathbf{L}$-*congruence* on $\mathbf{A}$ if θ is compatible w.r.t. all $f^{\mathbf{A}} \in F^{\mathbf{A}}$, $x \approx^{\mathbf{A}} y \leq \theta(x, y)$, and $\theta(x, y) \otimes (x \approx^{\mathbf{A}} x') \otimes (y \approx^{\mathbf{A}} y') \leq \theta(x', y')$ for all $x, x', y, y' \in A$. Clearly, for any $\mathbf{L}$-equivalence E on a set U, the relation ${}^1E = \{\langle x, y\rangle \mid E(x, y) = 1\}$ is an equivalence on U. For an $\mathbf{L}$-congruence θ on an $\mathbf{L}$-algebra $\mathbf{A}$, we define the *factor algebra* $\mathbf{A}/\theta = \langle A/\theta, \approx^{\mathbf{A}/\theta}, F^{\mathbf{A}/\theta} \rangle$ by $A/\theta = A/{}^1\theta$, $[x] \approx^{\mathbf{A}/\theta} [y] = \theta(x, y)$ (where $[x] = \{y \in A \mid \langle x, y\rangle \in {}^1\theta\}$), and each n-ary $f^{\mathbf{A}/\theta} \in F^{\mathbf{A}/\theta}$ is defined by $f^{\mathbf{A}/\theta}([x_1], \ldots, [x_n]) = [f^{\mathbf{A}}(x_1, \ldots, x_n)]$. $\mathbf{A}/\theta$ is well defined: If $[x] = [x']$, $[y] = [y']$, then $\theta(x, y) = \theta(x', x) \otimes \theta(x, y) \otimes \theta(y, y') \leq \theta(x', y')$ and similarly $\theta(x', y') \leq \theta(x, y)$, i.e. $\theta(x, y) = \theta(x', y')$, whence $\approx^{\mathbf{A}/\theta}$ is defined correctly. If $[x_1] = [y_1], \ldots, [x_n] = [y_n]$, then $1 = \theta(x_1, y_1) \otimes \cdots \otimes \theta(x_n, y_n) \leq \theta(f^{\mathbf{A}}(x_1, \ldots, x_n), f^{\mathbf{A}}(y_1, \ldots, y_n))$ which implies that $f^{\mathbf{A}/\theta}$ is correctly defined. Furthermore, $([x_1] \approx^{\mathbf{A}/\theta} [y_1]) \otimes \cdots \otimes ([x_n] \approx^{\mathbf{A}/\theta} [y_n]) = \theta(x_1, y_1) \otimes \cdots \otimes \theta(x_n, y_n) \leq \theta(f^{\mathbf{A}}(x_1, \ldots, x_n), f^{\mathbf{A}}(y_1, \ldots, y_n)) = f^{\mathbf{A}/\theta}([x_1], \ldots, [x_n]) \approx^{\mathbf{A}/\theta} f^{\mathbf{A}/\theta}([y_1], \ldots, [y_n])$, i.e. each operation on $\mathbf{A}/\theta$ is compatible w.r.t. $\approx^{\mathbf{A}/\theta}$. An $\mathbf{L}$-congruence θ on $\mathbf{A}$ is called *fully invariant* if for any homomorphism h from $\mathbf{A}$ into $\mathbf{A}$ we have $\theta(x, y) \leq \theta(h(x), h(y))$.

We denote $\mathbf{T}(X) = \langle T(X), \approx^{\mathbf{T}(X)}, F\rangle$ the $\mathbf{L}$-algebra such that $\langle T(X), F\rangle$ is the usual term algebra and $\approx^{\mathbf{T}(X)}$ is the identity $\mathrm{id}_{T(X)}$ on $T(X)$ (considered as an $\mathbf{L}$-relation). For a class $\mathcal{K}$ of $\mathbf{L}$-algebras of the same type we define an $\mathbf{L}$-relation $\mathrm{Id}_{\mathcal{K}}$ on $T(X)$ by $\mathrm{Id}_{\mathcal{K}}(p, q) = \| p \approx q \|_{\mathcal{K}}$. An $\mathbf{L}$-set Σ of identities is called an $\mathbf{L}$-*equational theory* iff $\Sigma = \mathrm{Id}_{\mathcal{K}}$ for some $\mathcal{K}$.

Lemma 2 $\mathrm{Id}_{\mathcal{K}}$ *is a fully invariant* $\mathbf{L}$*-congruence on* $\mathbf{T}(X)$.

Proof. Reflexivity and symmetry of $\mathrm{Id}_{\mathcal{K}}$ is obvious. Transitivity follows by $\mathrm{Id}_{\mathcal{K}}(p, q) \otimes \mathrm{Id}_{\mathcal{K}}(q, r) = \bigwedge_{\mathbf{A}\in\mathcal{K}} \| p \approx q \|_{\mathbf{A}} \otimes \bigwedge_{\mathbf{A}\in\mathcal{K}} \| q \approx r \|_{\mathbf{A}} \leq \bigwedge_{\mathbf{A}\in\mathcal{K}} (\| p \approx q \|_{\mathbf{A}} \otimes \| q \approx r \|_{\mathbf{A}}) = \bigwedge_{\mathbf{A}\in K} (\bigwedge_{v:X\to A} \| p \|_{\mathbf{A},v} \approx^{\mathbf{A}} \| q \|_{\mathbf{A},v}) \otimes (\bigwedge_{v:X\to A} \| q \|_{\mathbf{A},v} \approx^{\mathbf{A}} \| r \|_{\mathbf{A},v}) \leq \bigwedge_{\mathbf{A}\in K} (\bigwedge_{v:X\to A} (\| p \|_{\mathbf{A},v} \approx^{\mathbf{A}} \| q \|_{\mathbf{A},v}) \otimes (\| q \|_{\mathbf{A},v} \approx^{\mathbf{A}} \| r \|_{\mathbf{A},v})) \leq \bigwedge_{\mathbf{A}\in K} \bigwedge_{v:X\to A} (\| p \|_{\mathbf{A},v} \approx^{\mathbf{A}} \| r \|_{\mathbf{A},v}) = \mathrm{Id}_{\mathcal{K}}(p, r)$. Since $\approx^{\mathbf{T}(X)}$ is the identity relation, compatibility of any $f^{\mathbf{T}(X)}$ w.r.t. $\mathrm{Id}_{\mathcal{K}}$ w.r.t. is trivial. Furthermore, $\mathrm{Id}_{\mathcal{K}}(f(p_1, \ldots, p_n), f(q_1, \ldots, q_n)) = \bigwedge_{\mathbf{A}\in\mathcal{K}} \bigwedge_{v:X\to A} \| f(p_1, \ldots, p_n) \|_{\mathbf{A},v} \approx^{\mathbf{A}} \| f(q_1, \ldots, q_n) \|_{\mathbf{A},v}$, thus $f^{\mathbf{T}(X)}$ is compatible w.r.t. $\approx^{\mathbf{T}(X)}$ iff for any $\mathbf{A} \in \mathcal{K}$ and any $v : X \to A$ we have $\otimes_{i=1}^{n} \mathrm{Id}_{\mathcal{K}}(p_i, q_i) \leq \| f(p_1, \ldots, p_n) \|_{\mathbf{A},v} \approx^{\mathbf{A}} \| f(q_1, \ldots, q_n) \|_{\mathbf{A},v}$. The last inequality follows from compatibility of $f^{\mathbf{A}}$: $\otimes_{i=1}^{n} \mathrm{Id}_{\mathcal{K}}(p_i, q_i) \leq \otimes_{i=1}^{n} (\| p_i \|_{\mathbf{A},v} \approx^{\mathbf{A}} \| q_i \|_{\mathbf{A},v}) \leq f^{\mathbf{A}}(\| p_1 \|_{\mathbf{A},v}, \ldots, \| p_n \|_{\mathbf{A},v}) \approx^{\mathbf{A}} f^{\mathbf{A}}(\| q_1 \|_{\mathbf{A},v}, \ldots, \| q_n \|_{\mathbf{A},v})$. Finally, we check that $\mathrm{Id}_{\mathcal{K}}$ is fully invariant: Let $h : T(X) \to T(X)$ be a homomorphism. We have to check that $\mathrm{Id}_{\mathcal{K}}(p(x_1, \ldots, x_n), q(x_1, \ldots, x_n)) \leq \mathrm{Id}_{\mathcal{K}}(h(p(x_1, \ldots, x_n)), h(q(x_1, \ldots, x_n)))$. Since $h(r(x_1, \ldots, x_n)) = r(h(x_1), \ldots, h(x_n))$ is valid for any term $r \in T(X)$, the last inequality is true if for any $\mathbf{A} \in \mathcal{K}$ and any $v : X \to A$ there is some $v' : X \to A$ such that $\| p \approx q \|_{\mathbf{A},v'} \leq \| p(h(x_1),\ldots,h(x_n)) \approx q(h(x_1),\ldots,h(x_n)) \|_{\mathbf{A},v}$. A moment reflection shows that one may take any v' such that for any $i = 1,\ldots,n$, $v'(x_i) = \| h(x_i) \|_{\mathbf{A},v}$. □

Lemma 3 *For a fully invariant* **L***-congruence* θ *on* $\mathbf{T}(X)$ *we have* $\| p \approx q \|_{\mathbf{T(X)}/\theta} = \theta(p,q)$ *for every* $p,q \in T(X)$.

Proof. "$\leq$": For $v : X \to T(X)/\theta$ such that $v(x) = [x]$ $(x \in X)$ we have $\| p \approx q \|_{\mathbf{T(X)}/\theta} \leq \| p \|_{\mathbf{T(X)}/\theta,v} \approx^{\mathbf{T}(X)} \| q \|_{\mathbf{T(X)}/\theta,v} = [p] \approx^{\mathbf{T}(X)/\theta} [q] = \theta(p,q)$.

"$\geq$": Take any $v : X \to T(X)/\theta$, let $v(x_1) = [r_1],\ldots,v(x_n) = [r_n]$. Consider any homomorphism $h : T(X) \to T(X)$ such that $h(x_i) = r_i$. We have $\theta(p,q) \leq \theta(h(p),h(q)) = \theta(p(h(x_1),\ldots,h(x_n)),q(h(x_1),\ldots,h(x_n))) = \theta(p(r_1,\ldots,r_n),q(r_1,\ldots,r_n)) = \| p \|_{\mathbf{T(X)}/\theta,v} \approx^{\mathbf{T}(X)/\theta} \| q \|_{\mathbf{T(X)}/\theta,v}$. Since v is chosen arbitrarily, we conclude $\theta(p,q) \leq \| p \approx q \|_{\mathbf{T(X)}/\theta}$. □

Corollary 4 *Let* Σ *be an* **L***-set of identities, put* $\theta(p,q) = \Sigma(p \approx q)$*. Then* θ *is a fully invariant* **L***-congruence on* $\mathbf{T}(X)$ *iff* Σ *is an* **L***-equational theory.*

Proof. The "$\Leftarrow$" part is Lemma 2, the "$\Rightarrow$" part follows from Lemma 3 by putting $\mathcal{K} = \{\mathbf{T}(X)/\theta\}$. □

For an **L**-set Σ of identities denote by $\theta(\Sigma)$ the least fully invariant **L**-congruence on $\mathbf{T}(X)$ such that $\Sigma(p \approx q) \leq (\theta(\Sigma))(p,q)$ (its existence follows from the fact that the set of all fully invariant **L**-congruences on $\mathbf{T}(X)$ forms a complete lattice w.r.t. $\subseteq$).

Lemma 5 *For any* **L***-set* Σ *of identities, and every terms* $p,q \in T(X)$ *we have* $\Sigma \models p \approx q \quad = \quad (\theta(\Sigma))(p,q)$.

Proof. "$\leq$": We have $\Sigma(p \approx q) \leq (\theta(\Sigma))(p,q) = \| p \approx q \|_{\mathbf{T(X)}/\theta(\mathbf{\Sigma})}$, whence $\Sigma \models p \approx q \leq (\theta(\Sigma))(p,q)$.

"$\geq$": Take any $\mathbf{A}$ such that $\Sigma(r \approx s) \leq \| r \approx s \|_{\mathbf{A}}$. By Lemma 2, $\mathrm{Id}_{\{\mathbf{A}\}}$ is a fully invariant **L**-congruence on $\mathbf{T}(X)$ containing Σ. Therefore, $(\theta(\Sigma))(p,q) \leq \mathrm{Id}_{\{\mathbf{A}\}}(p,q) = \| p \approx q \|_{\mathbf{A}}$. The conclusion readily follows. □

The semantic consequence **L**-relation $\models$ defines naturally a fuzzy closure operator $\mathrm{C}^{\mathrm{sem}}$ on $T(X) \times T(X)$ (i.e. a mapping $\mathrm{C}^{\mathrm{sem}} : L^{T(X)\times T(X)} \to L^{T(X)\times T(X)}$ satisfying $\Sigma \subseteq \mathrm{C}^{\mathrm{sem}}(\Sigma)$, $\mathrm{C}^{\mathrm{sem}}(\Sigma_1) \subseteq \mathrm{C}^{\mathrm{sem}}(\Sigma_2)$ whenever $\Sigma_1 \subseteq \Sigma_2$, and $\mathrm{C}^{\mathrm{sem}}(\mathrm{C}^{\mathrm{sem}}(\Sigma)) = \mathrm{C}^{\mathrm{sem}}(\Sigma)$, see [1]) by $(\mathrm{C}^{\mathrm{sem}}(\Sigma))(p,q) \quad = \quad \Sigma \models p \approx q$.

We define a syntactic consequence operator $\mathrm{C}^{\mathrm{syn}}$ by letting $\mathrm{C}^{\mathrm{syn}}(\Sigma)$ be the least **L**-set in $T(X) \times T(X)$ such that $\Sigma \subseteq \mathrm{C}^{\mathrm{syn}}(\Sigma)$ and

(RE') $(\mathrm{C}^{\mathrm{syn}}(\Sigma))(p,p) = 1$;

(SY') $(\mathrm{C}^{\mathrm{syn}}(\Sigma))(p,q) \leq (\mathrm{C}^{\mathrm{syn}}(\Sigma))(q,p)$;

(TR') $(\mathrm{C}^{\mathrm{syn}}(\Sigma))(p,q) \otimes (\mathrm{C}^{\mathrm{syn}}(\Sigma))(q,r) \leq (\mathrm{C}^{\mathrm{syn}}(\Sigma))(p,r)$;

(REP') $(\mathrm{C}^{\mathrm{syn}}(\Sigma))(p,q) \leq (\mathrm{C}^{\mathrm{syn}}(\Sigma))(r,s)$ where where r is any term containing p as a subterm and s results from r by replacing one occurence of p by q;

(SUB') $(\mathrm{C}^{\mathrm{syn}}(\Sigma))(p,q) \leq (\mathrm{C}^{\mathrm{syn}}(\Sigma))(p(x/r), q(x,r))$

holds for any $p,q,r \in T(X)$. Note that the existence of $\mathrm{C}^{\mathrm{syn}}$ follows from the fact that the set of all **L**-sets in $T(X) \times T(X)$ which contain Σ and satisfy (RE')–(SUB') is not empty and is closed w.r.t. arbitrary intersections.

Lemma 6 *For any* **L***-set* Σ *of identities and every* $p,q \in T(X)$ *we have* $(\mathrm{C}^{\mathrm{sem}}(\Sigma))(p,q) = (\mathrm{C}^{\mathrm{syn}}(\Sigma))(p,q)$.

Proof. By Lemma 5, we have to prove $\mathrm{C}^{\mathrm{syn}}(\Sigma) = \theta(\Sigma)$. By (RE')–(TR'), $\mathrm{C}^{\mathrm{syn}}(\Sigma)$ is an **L**-equivalence on $T(X)$ which contains Σ. To show that $\mathrm{C}^{\mathrm{syn}}(\Sigma)$ is an **L**-congruence on $\mathbf{T}(X)$, take any n-ary $f \in F$, any $p_1, q_1, \ldots, p_n, q_n \in T(X)$, and put $s_i = f(q_1, \ldots, q_i, p_{i+1}, \ldots, p_n)$ $(i = 0, \ldots, n)$. By (REP'), $(\mathrm{C}^{\mathrm{syn}}(\Sigma))(p_i, q_i) \leq (\mathrm{C}^{\mathrm{syn}}(\Sigma))(s_i, s_{i+1})$, therefore $(\mathrm{C}^{\mathrm{syn}}(\Sigma))(p_1, q_1) \otimes \cdots \otimes (\mathrm{C}^{\mathrm{syn}}(\Sigma))(p_n, q_n) \leq (\mathrm{C}^{\mathrm{syn}}(\Sigma))(s_1, s_2) \otimes \cdots \otimes (\mathrm{C}^{\mathrm{syn}}(\Sigma))(s_{n-1}, s_n) \leq (\mathrm{C}^{\mathrm{syn}}(\Sigma))(s_1, s_n) = (\mathrm{C}^{\mathrm{syn}}(\Sigma))(f(p_1, \ldots, p_n), f(q_1, \ldots, q_n))$, i.e. $\mathrm{C}^{\mathrm{syn}}(\Sigma)$ is an **L**-congruence. We show that $\mathrm{C}^{\mathrm{syn}}(\Sigma)$ is fully invariant. Take any homomorphism $h : T(X) \to T(X)$ and any variables $y_1, \ldots, y_n$ so that no y_i occurs in x_j or $h(x_j)$, $j = 1, \ldots, n$. (SUB') yields

$$
\begin{aligned}
& (\mathrm{C}^{\mathrm{syn}}(\Sigma)(p(x_1, \ldots, x_n), q(x_1, \ldots, x_n)) = \\
= \; & (\mathrm{C}^{\mathrm{syn}}(\Sigma)(p(y_1, \ldots, y_n), q(y_1, \ldots, y_n)) \leq \\
\leq \; & (\mathrm{C}^{\mathrm{syn}}(\Sigma)(p(h(x_1), \ldots, y_n), q(h(x_1), \ldots, y_n)) \leq \cdots \leq \\
\leq \; & (\mathrm{C}^{\mathrm{syn}}(\Sigma)(p(h(x_1), \ldots, h(x_n)), q(h(x_1), \ldots, h(x_n))) = \\
= \; & (\mathrm{C}^{\mathrm{syn}}(\Sigma)(h(p(x_1, \ldots, x_n)), h(q(x_1, \ldots, x_n))),
\end{aligned}
$$

completing the proof. □

Lemma 7 *For any* **L***-set* Σ *of identities and every* $p,q \in T(X)$ *we have* $\Sigma \vdash p \approx q = (\mathrm{C}^{\mathrm{syn}}(\Sigma))(p,q)$.

Proof. "$\leq$": Clearly, it suffices to check that if $\langle p_i \approx q_i, a_i \rangle$ is a member of some **L**-valuated proof, then $a_i \leq (\mathrm{C}^{\mathrm{syn}}(\Sigma))(p_i, q_i)$. This is obvious if $a_i = \Sigma(p_i \approx q_i)$. Otherwise (i.e. $\langle p_i \approx q_i, a_i \rangle$ is obtained by some inference rule), proceed by induction over i and suppose that the assertion is true for $j < i$. The following inference rules could have been used:

(RE): $p_i = q_i$ and $a_i = 1 = (\mathrm{C}^{\mathrm{syn}}(\Sigma))(p_i, q_i)$ by (RE');

(SY): $p_i = q_j$, $q_i = p_j$, and $a_i = a_j$ for some $j < i$ and thus $a_i \leq (\mathrm{C}^{\mathrm{syn}}(\Sigma))(p_j, q_j) \leq\leq (\mathrm{C}^{\mathrm{syn}}(\Sigma))(p_i, q_i)$ by (SY');

(TR): $p_i = p_j$, $q_j = p_k$, $q_k = q_i$, and $a_i = a_j \otimes a_k$ for some $j, k < i$ and thus $a_i \leq (\mathrm{C}^{\mathrm{syn}}(\Sigma))(p_j, q_j) \otimes (\mathrm{C}^{\mathrm{syn}}(\Sigma))(p_k, q_k) \leq (\mathrm{C}^{\mathrm{syn}}(\Sigma))(p_i, q_i)$ by (TR');

(REP): q_i is obtained from p_i by replacement of some occurrence of p_j by q_j, and $a_i = a_j$ for some $j < i$, whence $a_i = (\mathrm{C}^{\mathrm{syn}}(\Sigma))(p_j, q_j) \leq (\mathrm{C}^{\mathrm{syn}}(\Sigma))(p_i, q_i)$ by (REP');

(SUB): $p_i = p_j(x/r)$, $q_i = q_j(x/r)$, and $a_i = a_j$ for some $r \in T(X)$ and $j < i$, thus $a_i = (\mathrm{C}^{\mathrm{syn}}(\Sigma))(p_j, q_j) \leq (\mathrm{C}^{\mathrm{syn}}(\Sigma))(p_i, q_i)$ by (SUB').

"$\geq$": It suffices to prove that the **L**-set D in $T(X) \times T(X)$ defined by $D(p, q) = \Sigma \vdash p \approx q$ contains Σ and satisfies (RE')–(SUB') (the inequality then follows from the fact that $\mathrm{C}^{\mathrm{syn}}(\Sigma)$ is the least **L**-set with these properties). Since $\langle p \approx q, \Sigma(p \approx q) \rangle$ is a proof from Σ, we have $\Sigma \subseteq D$.

(RE') follows from the fact that $\langle p \approx p, 1 \rangle$ is a proof.

(SY') follows from the fact that if $\ldots, \langle p \approx q, a \rangle$ is a proof from Σ then $\ldots, \langle p \approx q, a \rangle, \langle q \approx p, a \rangle$ is as well.

(TR'): Let $u_i, \langle p \approx q, a_i \rangle$ $(i \in I)$ and $v_j, \langle q \approx r, b_j \rangle$ $(j \in J)$ be all proofs of $p \approx q$ and $q \approx r$ from Σ, respectively. Since each $u_i, v_j, \langle p \approx q, a_i \rangle, \langle q \approx r, b_j \rangle, \langle p \approx r, a_i \otimes a_i \otimes b_j \rangle$ is a proof of $p \approx r$ from Σ, we have $D(p, q) \otimes D(q, r) = (\bigvee_{i \in I} a_i) \otimes (\bigvee_{j \in J} b_j) = \bigvee_{i \in I}(a_i \otimes (\bigvee_{j \in J} b_j)) = \bigvee_{i \in I, j \in J} a_i \otimes b_j \leq \bigvee\{c \mid w, \langle p \approx r, c \rangle \text{ is a proof from } \Sigma\} = D(p, r)$.

(REP') and (SUB) follows from the fact that if $\ldots, \langle p \approx q, a \rangle$ is a proof from Σ and $r \approx s$ is obtained from $p \approx q$ by replacement or substitution then $\ldots, \langle p \approx q, a \rangle, \langle r \approx s, a \rangle$ is a proof from Σ as well. □

Theorem 1 is now a direct consequence of Lemma 6 and Lemma 7.

Remark 1 To the notion of an **L**-algebra: there are natural examples. For instance, let U be a set equipped with an **L**-equivalence $\approx^U$. Let be $A = S(U)$ be the set of all permutations of U. The triple $\mathbf{A} = \langle A, \approx^{\mathbf{A}}, \circ^{\mathbf{A}} \rangle$ where $\pi \approx^{\mathbf{A}} \sigma = \bigwedge_{u,v}(u \approx^U v) \to (\pi(u) \approx^U \sigma(v))$ and $\circ^{\mathbf{A}}$ denotes the composition of permutations, is an **L**-algebra. Indeed, one easily verifies that $\approx^{\mathbf{A}}$ is an **L**-equivalence on A and that $\circ^{\mathbf{A}}$ is compatible w.r.t. $\approx^{\mathbf{A}}$. For details as well as for further information about **L**-algebras we refer to [1].

Remark 2 The provability degree may be stictly greater than the value of any proof. Indeed, let $F = \{\circ\}$, $\circ$ be binary, denote by x^n the n-th power of x w.r.t. $\circ$, i.e. $x^3 = (x \circ x) \circ x$ etc. Let **L** be the standard Łukasiewicz algebra on $[0, 1]$. Define Σ by $\Sigma(x \circ x \approx x) = 1$, $\Sigma(x^n \approx y^n) = 1 - 1/n$, and $\Sigma(p \approx q) = 0$ otherwise. Clearly, $\langle x \circ x \approx x, 1 \rangle, \langle y \circ y \approx y, 1 \rangle, \langle x^n \approx y^n, 1 - 1/n \rangle, \langle x^{n-1} \approx y^n, 1 - 1/n \rangle, \langle x^{n-1} \approx y^{n-1}, 1 - 1/n \rangle$,..., $\langle x \approx y, 1 - 1/n \rangle$ is an **L**-valuated proof of $x \approx y$ from Σ for any n. Therefore, $\Sigma \vdash x \approx y = 1$. On the other hand, there is no proof of $x \approx y$ from Σ the value of which is 1: Such a proof must not use any $\langle x^n \approx y^n, 1 - 1/n \rangle$, and therefore uses only $\langle x \circ x \approx x, 1 \rangle$ which is impossible (any **L**-algebra satisfying $x \circ x \approx x$ in degree 1 but $x \approx y$ in degree less than 1, e.g. an idempotant monoid with at least two elements and the classical identity as the **L**-equality, serves as a counterexample).

Remark 3 As observed by Pavelka [10], we cannot have graded style completeness for arbitrary complete residuated lattice even in the case of propositional logic (the less so for the first-order case [5, 8]). However, since (RE)–(SUB) can be used in Pavelka style first-oder fuzzy logic as derived rules (more precisely: derived rules in first-order fuzzy logic with the usual inference rules (for details see [5, 8]) where the predicate symbol $\approx$ is confined (in an obvious sense) by axioms of reflexivity, symmetry, transitivity, and compatibility), our result implies that the equational fragment (i.e. restriction to formulas of the form of identities) is completely axiomatizable (in the graded style) in first-order fuzzy logic using any complete residuated lattice as the structure of truth values.

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