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Lattices of fixed points of fuzzy Galois connections

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Abstract. We give a characterization of the fixed points and of the lattices of fixed points of fuzzy Galois connections. It is shown that fixed points are naturally interpreted as concepts in the sense of traditional logic.

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1 Introduction

The notion of Galois connection has been generalized from the point of view of fuzzy set theory in [2]. The main aim of the present paper is to characterize the lattices of fixed points of fuzzy Galois connections.

A *residuated lattice* [12, 11, 8, 7] is an algebra $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ where

(i) $\langle L, \wedge, \vee, 0, 1\rangle$ is a lattice with the least element 0 and the greatest element 1,

(ii) $\langle L, \otimes, 1\rangle$ is a commutative monoid, i.e. $\otimes$ is associative, commutative, and the identity $x \otimes 1 = x$ holds,

(iii) $\otimes$ and $\rightarrow$ satisfy the adjointness property, i.e.

$$x \leq y \rightarrow z \quad \text{iff} \quad x \otimes y \leq z$$

holds for each $x, y, z \in L$ ($\leq$ denotes the lattice ordering).

Residuated lattices play the role of the structures of truth values in fuzzy logic and fuzzy set theory [7, 8, 9]. A semantically complete first-order many-valued logic with semantics defined over complete residuated lattices is described in [8]. Thorough the paper **L** denotes a complete residuated lattice. For the properties of complete residuated lattices needed in the sequel we refer to [2].

Recall that an **L**-*set* (or fuzzy set) [5, 6, 14] in a universe set X is any function $A : X \rightarrow L$. The value $A(x)$ is interpreted as the truth value of "x is element of A". Similarly, a fuzzy relation between X and Y is any function $I : X \times Y \rightarrow L$. By $\{ {}^{a}/x \}$ (where $a \in L$, $x \in X$) it is meant a fuzzy set given by $\{ {}^{a}/x \}(x) = a$ and $\{ {}^{a}/x \}(x') = 0$ for $x' \in X$, $x' \neq x$. In this perspective, classical sets (relations) are identified with **2**-sets (**2**-relations) where **2** denotes the two-element Boolean algebra of classical logic. The set of all **L**-sets in a given universe X will be denoted by L^X. For $A_1, A_2 \in L^X$, the *subsethood degree* [5] of A_1 in A_2 is defined by $Subs(A_1, A_2) = \bigwedge_{x \in X}(A_1(x) \rightarrow A_2(x))$. We write $A_1 \subseteq A_2$ for $Subs(A_1, A_2) = 1$.

Definition 1 ([2]) *An* **L**-Galois connection *(*fuzzy Galois connection*) between the sets X and Y is a pair $\langle ^{\uparrow}, ^{\downarrow} \rangle$ of mappings $^{\uparrow} : L^X \rightarrow L^Y$, $^{\downarrow} : L^Y \rightarrow L^X$, satisfying*

$$\begin{aligned} Subs(A_1, A_2) &\leq Subs(A_2^{\uparrow}, A_1^{\uparrow}) && (1)\\ Subs(B_1, B_2) &\leq Subs(B_2^{\downarrow}, B_1^{\downarrow}) && (2)\\ A &\subseteq (A^{\uparrow})^{\downarrow} && (3)\\ B &\subseteq (B^{\downarrow})^{\uparrow} . && (4) \end{aligned}$$

for every $A, A_1, A_2 \in L^X$, $B, B_1, B_2 \in L^Y$.

Note that for $\mathbf{L} = \mathbf{2}$ we get the classical notion of Galois connection [3, 10]. The lattices of fixed points of Galois connections have been characterized in [13]. In Section 2 we provide a generalization of that characterization for **L** being any complete residuated lattice.

The following theorem shows that **L**-Galois connections are precisely the mappings induced by binary **L**-relations.

Theorem 2 ([2]) *For a binary* **L***-relation* $I \in L^{X\times Y}$ *denote* $\langle {}^{\uparrow_I}, {}^{\downarrow_I} \rangle$ *the mappings defined by*

$$A^{\uparrow_I}(y) = \bigwedge_{x\in X} (A(x) \rightarrow I(x,y)) \quad \text{for} \quad y \in Y \tag{5}$$

$$B^{\downarrow_I}(x) = \bigwedge_{y\in Y} (B(y) \rightarrow I(x,y)) \quad \text{for} \quad x \in X \tag{6}$$

for any $A \in L^X$, $B \in L^Y$. *For an* **L***-Galois connection* $\langle {}^{\uparrow}, {}^{\downarrow} \rangle$ *between* X *and* Y *denote* $I_{\langle \uparrow, \downarrow \rangle}$ *the binary* **L***-relation between* X *and* Y *defined by*

$I_{\langle \uparrow, \downarrow \rangle}(g,m) = \{\,{}^1/g\}^{\uparrow}(m)$. *Then* $\langle {}^{\uparrow_I}, {}^{\downarrow_I} \rangle$ *is an* **L***-Galois connection and it holds*

$$\langle {}^{\uparrow}, {}^{\downarrow} \rangle = \langle {}^{\uparrow_{I_{\langle \uparrow, \downarrow \rangle}}}, {}^{\downarrow_{I_{\langle \uparrow, \downarrow \rangle}}} \rangle \quad \text{and} \quad I = I_{\langle \uparrow_I, \downarrow_I \rangle}.$$

2 Lattices of fixed points

In this section, we denote by $\langle {}^{\uparrow}, {}^{\downarrow} \rangle$ and I the corresponding **L**-Galois connection and **L**-relation between X and Y, respectively, omitting thus the indicies.

Definition 3 *A* fixed point *of* $\langle {}^{\uparrow}, {}^{\downarrow} \rangle$ *is a pair* $\langle A, B \rangle \in L^X \times L^Y$ *such that* $A^{\uparrow} = B$ *and* $B^{\downarrow} = A$.

Therefore, if $\langle A, B \rangle$ is a fixed point then $A^{\uparrow\downarrow} = A$ and $B^{\downarrow\uparrow} = B$. We are going to show that fixed points of $\langle {}^{\uparrow}, {}^{\downarrow} \rangle$ correspond to the maximal rectangles contained in I. For $A \in L^X$, $B \in L^Y$, denote by $A \otimes B$ the **L**-set in $X \times Y$ defined by $(A \otimes B)(x,y) = A(x) \otimes B(y)$. Call a *rectangle* any pair $\langle A, B \rangle \in L^X \times L^Y$. There is a naturally defined ordering $\leq$ defined on the set of all rectangles by $\langle A_1, B_1 \rangle \leq \langle A_2, B_2 \rangle$ iff for all $x \in X$, $y \in Y$ it holds $A_1(x) \leq A_2(x)$ and $B_1(y) \leq B_2(y)$. A rectangle $\langle A, B \rangle$ is said to be *contained* in I if $A \otimes B \subseteq I$. The following theorem generalizes the observation of the classical case stating that fixed points are just maximal rectangles of I which are filled with 1's (if we consider the two-valued relation I as a matrix-table of 0's and 1's).

Theorem 4 *For each* $A \in L^X$, $B \in L^Y$ *it holds that* $\langle A, B \rangle$ *is a fixed point of* $\langle {}^{\uparrow}, {}^{\downarrow} \rangle$ *iff it is a maximal rectangle contained in* I.

Proof. Let $\langle A, B \rangle$ be a fixed point. If it were not maximal, there would be $\langle A', B' \rangle \in L^X \times L^Y$ such that $\langle A, B \rangle < \langle A', B' \rangle$. Hence, there exists an

$x \in X$ such that $A(x) < A'(x)$ or $y \in Y$ such that $B(y) < B'(y)$. Suppose the former, i.e. $A(x) < A'(x)$ for some $x \in X$ (the latter may be handled analogously). By assumption, $B(y) \le B'(y)$ holds for all $y \in Y$, therefore $A'(x) \otimes B(y) \le A'(x) \otimes B'(y) \le I(x,y)$, and thus $A'(x) \le B(y) \to I(x,y)$ holds for each $y \in Y$. We conclude

$$A(x) < A'(x) \le \bigwedge_{y \in Y} (B(y) \to I(x,y)) = B^{\downarrow}(x),$$

a contradiction to $A = B^{\downarrow}$ ($\langle A, B\rangle$ is a fixed point).

Conversely, let $\langle A, B\rangle$ be a maximal rectangle contained in I. We have to show $A = B^{\downarrow}$ and $B = A^{\uparrow}$. From $A(x) \to B(y) \le I(x,y)$ it follows $A(x) \le B(y) \to I(x,y)$ for all $x \in X$, $y \in Y$, i.e. $A(x) \le B^{\downarrow}(x) = \bigwedge_{y \in Y}(B(y) \to I(x,y))$, thus $A \subseteq B^{\downarrow}$. $\left\langle B^{\downarrow}, B\right\rangle$ is contained in I, since $B^{\downarrow}(x) \otimes B(y) \le I(x,y)$ is equivalent to $B^{\downarrow}(x) \le B(y) \to I(x,y)$ which holds evidently. If $A \ne B^{\downarrow}$, i.e. $A(x) < B^{\downarrow}(x)$ for some $x \in X$, then $\langle A, B\rangle < \left\langle B^{\downarrow}, B\right\rangle$, a contradiction to the maximality of $\langle A, B\rangle$ among the rectangles contained in I. The condition $B = A^{\uparrow}$ may be shown analogously. $\square$

Denote $\mathcal{L}(X, Y, I) = \{\langle A, B\rangle \in L^X \times L^Y \mid A^{\uparrow} = B,\, B^{\downarrow} = A\}$ the set of all fixed points of $\langle {}^{\uparrow}, {}^{\downarrow}\rangle$. Clearly, for $\langle A_1, B_1\rangle, \langle A_2, B_2\rangle \in \mathcal{L}(G, M, I)$ it holds $A_1 \subseteq A_2$ iff $B_1 \supseteq B_2$. Introduce therefore the partial order $\le$ on $\mathcal{L}(X, Y, I)$ by

$$\langle A_1, B_1\rangle \le \langle A_2, B_2\rangle \quad \text{iff} \quad A_1 \subseteq A_2 \quad (\text{ iff } B_1 \supseteq B_2).$$

The characterization of the ordered sets of fixed points is given by the following theorem.

Note that for a complete lattice $\mathbf{V}$, a subset $K \subseteq V$ is $\bigvee$-*dense* (supremally dense) in V ($\bigwedge$-*dense* (infimally dense) in V) if for each $v \in V$ there is $K' \subseteq K$ such that $v = \bigvee K'$ ($v = \bigwedge K'$).

Theorem 5 *Let $I \in L^{X \times Y}$. (1) $\mathcal{L}(X, Y, I)$ is a complete lattice in which infima and suprema can be described as follows:*

$$\bigwedge_{j \in J} \langle A_j, B_j\rangle = \left\langle \bigcap_{j \in J} A_j, (\bigcap_{j \in J} A_j)^{\uparrow} \right\rangle = \left\langle \bigcap_{j \in J} A_j, (\bigcup_{j \in J} B_j)^{\downarrow\uparrow} \right\rangle, \qquad (7)$$

$$\bigvee_{j \in J} \langle A_j, B_j\rangle = \left\langle (\bigcap_{j \in J} B_j)^{\downarrow}, \bigcap_{j \in J} B_j \right\rangle = \left\langle (\bigcup_{j \in J} A_j)^{\uparrow\downarrow}, \bigcap_{j \in J} B_j \right\rangle. \qquad (8)$$

(2) Moreover, a complete lattice $\mathbf{V} = \langle V, \leq \rangle$ is isomorphic to $\mathcal{L}(X,Y,I)$ iff there are mappings $\gamma : X \times L \to V$, $\mu : Y \times L \to V$, such that $\gamma(X \times L)$ is $\bigvee$-dense in $\mathbf{V}$, $\mu(Y \times L)$ is $\bigwedge$-dense in $\mathbf{V}$, and $a \otimes b \leq I(x,y)$ is equivalent to $\gamma(x,a) \leq \mu(y,b)$ for all $x \in X$, $y \in Y$, $a,b \in L$.

Proof. Part (1) of the assertion follows directly from the fact that $\mathcal{L}(X,Y,I)$ is the set of all closed points of the Galois connections $\langle {}^\uparrow, {}^\downarrow \rangle$ between the complete lattices $\langle L^X, \subseteq \rangle$ and $\langle L^Y, \subseteq \rangle$ [2, first Remark] and [10].

Part (2): Let $\mathcal{L}(X,Y,I)$ and $\mathbf{V}$ be isomorphic. We show the existence of γ, μ with the desired properties. It suffices to show the existence for $\mathbf{V} = \mathcal{L}(X,Y,I)$ because for the general case $\mathbf{V} \cong \mathcal{L}(X,Y,I)$ one can take take $\gamma \circ \varphi : X \times L \to V$, $\mu \circ \varphi : Y \times L \to V$, where φ is the isomorphism of $\mathcal{L}(X,Y,I)$ onto $\mathbf{V}$. Let then $\gamma : X \times L \to \mathcal{L}(X,Y,I)$, $\mu : Y \times L \to \mathcal{L}(X,Y,I)$ be defined by

$$\begin{aligned}\gamma(x,a) &= \left\langle \{ {}^a/x \}^{\uparrow\downarrow}, \{ {}^a/x \}^{\uparrow} \right\rangle, \\ \mu(y,b) &= \left\langle \left\{ {}^b/y \right\}^{\downarrow}, \left\{ {}^b/y \right\}^{\downarrow\uparrow} \right\rangle\end{aligned}$$

for every $x \in X$, $y \in Y$, $a,b \in L$. Since for each $\langle A,B \rangle \in \mathcal{L}(X,Y,I)$ it holds $A = \bigcup_{x \in X} \left\{ {}^{A(x)}/x \right\}$, and $B = \bigcup_{y \in Y} \left\{ {}^{B(y)}/y \right\}$, it follows from (7) and (8) that $\gamma(X,L)$ and $\mu(Y,L)$ are $\bigvee$-dense and $\bigwedge$-dense in $\mathcal{L}(X,Y,I)$, respectively. Furthermore, for any $x \in X$, $y \in Y$, $a,b \in L$ we have $\gamma(x,a) \leq \mu(y,b)$ iff $\{ {}^a/x \}^{\uparrow\downarrow} \subseteq \left\{ {}^b/y \right\}^{\downarrow}$ iff $\{ {}^a/x \}^{\uparrow} \supseteq \left\{ {}^b/y \right\}$ iff $\{ {}^a/x \}^{\uparrow}(y) \geq b$ iff $\bigwedge_{x' \in X} \{ {}^a/x \}(x') \to I(x',y) = \{ {}^a/x \}(x) \to I(x,y) \geq b$ iff $a \to I(x,y) \geq b$ iff $a \otimes b \leq I(x,y)$. Hence, γ and μ obey the required properties.

Conversely, let γ and μ with the above properties exist. We prove the assertion by showing that there are monotone mappings $\varphi : \mathcal{L}(X,Y,I) \to V$, $\psi : V \to \mathcal{L}(X,Y,I)$, such that $\varphi \circ \psi = id_{\mathcal{L}(X,Y,I)}$ and $\psi \circ \varphi = id_V$. We will need the following claims.

Claim A. *$\gamma(x, \bigvee_{j \in J} a_j) = \bigvee_{j \in J} \gamma(x, a_j)$, $\mu(y, \bigvee_{j \in J} a_j) = \bigwedge_{j \in J} \mu(y, a_j)$ for each $x \in X$, $y \in Y$, $\{a_j \mid j \in J\} \subseteq L$, i.e. $\gamma(x, \_) : L \to V$ are complete lattice $\bigvee$-morphisms and $\mu(y, \_) : L \to V$ are dual complete lattice $\bigwedge$-morphisms.*

Proof of Claim A. The $\bigwedge$-density of $\mu(Y,L)$ implies that $\gamma(x, \bigvee_{j \in J} a_j) = \bigwedge_{\langle y,b \rangle \in X} \mu(y,b)$ for some $X \subseteq Y \times L$. Hence, $\gamma(x, \bigvee_{j \in J} a_j) \leq \mu(y,b)$ which implies $(\bigvee_{j \in J} a_j) \otimes b \leq I(x,y)$, for each $\langle y,b \rangle \in X$. From $a_j \otimes b \leq (\bigvee_{j \in J} a_j) \otimes b$ we have $a_j \otimes b \leq I(x,y)$, i.e. $\gamma(x,a_j) \leq \mu(y,b)$ for every $j \in J$. This implies $\bigvee_{j \in J} \gamma(x,a_j) \leq \bigwedge_{\langle y,b \rangle \in X} \mu(y,b) = \gamma(x, \bigvee_{j \in J} a_j)$.

Conversely, the $\bigwedge$-density of $\mu(Y \times L)$ again implies the existence of some $X \subseteq Y \times L$ such that $\bigvee_{j \in J} \gamma(x, a_j) = \bigwedge_{\langle y,b \rangle \in X} \mu(y, b)$. That means that for each $j \in J$, $\langle y, b \rangle \in X$ we have $\gamma(x, a_j) \leq \mu(y, b)$, i.e. $a_j \otimes b \leq I(x, y)$. This implies $\bigvee_{j \in J}(a_j \otimes b) \leq I(x, y)$ and, by $\bigvee_{j \in J}(a_j \otimes b) = \bigvee_{j \in J} a_j \otimes b$, further $\bigvee_{j \in J} a_j \otimes b \leq I(x, y)$, i.e. $\gamma(x, \bigvee_{j \in J} a_j) \leq \mu(y, b)$ for each $\langle y, b \rangle \in X$, thus $\gamma(x, \bigvee_{j \in J} a_j) \leq \bigwedge_{\langle y,b \rangle \in X} \mu(y, b) = \bigvee_{j \in J} \gamma(x, a_j)$, proving $\gamma(x, \bigvee_{j \in J} a_j) = \bigvee_{j \in J} \gamma(x, a_j)$.

$\mu(y, \bigvee_{j \in J} a_j) = \bigwedge_{j \in J} \mu(y, a_j)$ may be proved analogously using the $\bigvee$-density of $\gamma(X \times L)$. *Q.E.D.*

Claim B. $I(x, y) = \bigvee_{\gamma(x,a) \leq \mu(y,b)} a \otimes b$.

Proof of Claim B. The inequality $I(x, y) \geq \bigvee_{\gamma(x,a) \leq \mu(y,b)} a \otimes b$ follows immediately. For $a = I(x, y)$, $b = 1$ we have $a \otimes b = I(x, y) \otimes 1 \leq I(x, y)$, hence $\gamma(x, I(x, y)) \leq \mu(y, 1)$, thus the equality holds. *Q.E.D.*

Define the mapping $\varphi : \mathcal{L}(X, Y, I) \to V$ by

$$\varphi(A, B) = \bigvee_{x \in X} \gamma(x, A(x)) \tag{9}$$

for each $\langle A, B \rangle \in \mathcal{L}(X, Y, I)$.

The monotonicity of φ follows from *Claim A*, (9) and the fact that $\langle A_1, B_1 \rangle \leq \langle A_2, B_2 \rangle$ implies $A_1(x) \leq A_2(x)$ for each $x \in X$.

We prove the existence of an inverse mapping ψ of φ. Define $\psi : V \to \mathcal{L}(X, Y, I)$ by

$$\psi(v) = \langle A, B \rangle, \quad \text{where } A(x) = \bigvee_{\gamma(x,a) \leq v} a, \; B(y) = \bigwedge_{\mu(y,b) \geq v} b \tag{10}$$

for each $v \in V$, and every $x \in X$, $y \in Y$. First, we show that for each $v \in V$, $\psi(v)$ is a fixed point of $\mathcal{L}(X, Y, I)$, i.e. $A^{\uparrow} = B$ and $B^{\downarrow} = A$. We show only $B^{\downarrow} = A$, the second case may be proved symmetrically. By Claim B we have

$$B^{\downarrow}(x) = \bigwedge_{y \in Y} B(y) \to I(x, y) = \bigwedge_{y \in Y} \Big(\bigwedge_{\mu(y,b) \geq v} b \to \bigvee_{\gamma(x,a) \leq \mu(y,b)} a \otimes b \Big).$$

We show $A(x) \leq B^{\downarrow}(x)$. We have

$$\bigvee_{\gamma(x,a) \leq v} a \otimes \bigwedge_{\mu(y,b) \geq v} b \leq \bigvee_{\gamma(x,a) \leq v \leq \mu(y,b)} a \otimes b \leq \bigvee_{\gamma(x,a) \leq \mu(y,b)} a \otimes b,$$

i.e.

$$A(x) = \bigvee_{\gamma(x,a) \leq v} a \leq \bigwedge_{\mu(y,b) \geq v} \to \bigvee_{\gamma(x,a) \leq \mu(y,b)} a \otimes b$$

holds for each $y \in Y$, hence also

$$A(x) = \bigvee_{\gamma(x,a)\leq v} a \leq \bigwedge_{y\in Y} (\bigwedge_{\mu(y,b)\geq v} b \to \bigvee_{\gamma(x,a)\leq \mu(y,b)} a \otimes b) = B^{\downarrow}(x),$$

holds. We now show the equality $A(x) = B^{\downarrow}(x)$ as follows. Suppose there is an $a \in L$ such that for each $y \in Y$ it holds

$$a \leq \bigvee_{\mu(y,b)\geq v} b \to I(x,y) \tag{11}$$

(i.e. a is a lower bound) and show that $a \leq \bigvee_{\gamma(x,a)\leq v} a = A(x)$ (i.e. $A(x)$ is the infimum, i.e. $B^{\downarrow}(x)$). (11) holds iff

$$a \otimes \bigvee_{\mu(y,b)\geq v} b \leq I(x,y),$$

i.e. by $a \otimes \bigvee_{\mu(y,b)\geq v} b = \bigvee_{\mu(y,b)\geq v}(a \otimes b)$ we get that for each b such that $\mu(y,b) \geq v$ it holds $a \otimes b \leq I(x,y)$. The last fact implies that for each b such that $\mu(y,b) \geq v$ it holds $\gamma(x,a) \leq \mu(y,b)$ which holds for each $y \in Y$. From the $\bigwedge$-density of $\mu(Y \times L)$ it follows that $v = \bigwedge_{v\leq\mu(y,b)} \mu(y,b)$, and hence $\gamma(x,a) \leq v$ which implies $a \leq \bigvee_{\gamma(x,a')\leq v} a' = A(x)$. We have proved $A = B^{\downarrow}$.

Next we show that $\varphi \circ \psi = id_{\mathcal{L}(X,Y,I)}$ and $\psi \circ \varphi = id_V$. For each $v \in V$ we have by Claim A and the $\bigvee$-density of $\gamma(X \times L)$

$$\begin{aligned}
\psi \circ \varphi(v) &= \varphi(A,B) = \bigvee_{x\in X} \gamma(x, A(x)) = \\
&= \bigvee_{x\in X} \gamma(x, \bigvee_{\gamma(x,a)\leq v} a) = \bigvee_{x\in X} \bigvee_{\gamma(x,a)\leq v} \gamma(x,a) = \\
&= \bigvee_{\gamma(x,a)\leq v} \gamma(x,a) = v,
\end{aligned}$$

i.e. $\psi \circ \varphi(v) = v$. Consider now $\varphi \circ \psi(A,B)$ for $\langle A,B\rangle \in \mathcal{L}(X,Y,I)$. First we show

$$\bigvee_{x\in X} \gamma(x, A(x)) = \bigwedge_{y\in Y} \mu(y, B(y)). \tag{12}$$

The inequality $\bigvee_{x\in X} \gamma(x,A(x)) \leq \bigwedge_{y\in Y} \mu(y,B(y))$ is inferred from the fact that for every $x \in X$, $y \in Y$ we have $\gamma(x,A(x)) \leq \mu(y,B(y))$ which follows from Claim B as here: $\gamma(x,A(x)) \leq \mu(y,B(y))$ holds iff $A(x)\otimes B(y) \leq I(x,y)$ iff $A(x) \leq B(y) \to I(x,y)$ which holds because of $A(x) = \bigwedge_{y'\in Y}(B(y') \to$

$I(x,y)) \leq B(y) \to I(x,y)$. To get the equality (12), denote $v = \varphi(A,B) = \bigvee_{x\in X} \gamma(x, A(x))$. We show that $\bigwedge_{y\in Y} \mu(y, B(y)) = v$. From the $\bigwedge$-density of $\mu(Y \times L)$ we have clearly $v = \bigwedge_{\mu(y,b)\geq v} \mu(y,b)$. We show that for each y, b such that $\mu(y,b) \geq v$ it holds $b \leq B(y)$. Indeed, if $\mu(y,b) \geq v$ then clearly $\mu(y,b) \geq \gamma(x, A(x))$ for all $x \in X$. If $b \leq B(y)$ is not the case then consider $b \vee B(y)$. For each $x \in X$ we have $\mu(y,b) \geq \gamma(x, A(x))$, $\mu(y, B(y)) \geq \gamma(x, A(x))$, hence, by Claim A, $\mu(y, b \vee B(y)) = \mu(y,b) \wedge \mu(y, B(y)) \geq \gamma(x, A(x))$. This implies $A(x) \otimes B(y) \leq A(x) \otimes (b \vee B(y)) \leq I(x,y)$, i.e. $b \vee B(y) \leq A(x) \to I(x,y)$ for each $x \in X$, i.e. $b \vee B(y) \leq \bigwedge_{x\in X} A(x) \to I(x,y) = B(y)$, i.e. $b \leq B(y)$, a contradiction. Furthermore, from $b \leq B(y)$ it follows by Claim A that $\mu(y, B(y)) \leq \mu(y,b)$. Thus, from $v \leq \mu(y,b)$ it follows $\mu(y, B(y)) \leq \mu(y,b)$. We conclude

$$v = \bigvee_{x\in X} \gamma(x, A(x)) \leq \bigwedge_{y\in Y} \mu(y, B(y)) \leq \bigwedge_{\mu(y,b)\geq v} \mu(y,b) = v,$$

i.e (12) holds. We therefore have

$$\begin{aligned} \varphi \circ \psi(A,B) &= \psi(\bigwedge_{y\in Y} \mu(y, B(y))) = \\ &= \Big\langle \{\langle x, \bigvee_{\gamma(x,a)\leq \bigwedge_{y\in Y} \mu(y,B(y))} a\rangle \mid x \in X\}, \\ &\quad \{\langle y, \bigvee_{\mu(y,b)\geq \bigwedge_{y\in Y} \mu(y,B(y))} b\rangle \mid y \in Y\} \Big\rangle. \end{aligned}$$

As $\varphi \circ \psi(A,B) \in \mathcal{L}(X,Y,I)$, it suffices to show that

$$\bigvee_{\gamma(x,a)\leq \bigwedge_{y\in Y} \mu(y,B(y))} a = A(x).$$

From (12) we have $\gamma(x, A(x)) \leq \bigwedge_{y\in Y} \mu(y, B(y))$ and therefore

$$\bigvee_{\gamma(x,a)\leq \bigwedge_{y\in Y} \mu(y,B(y))} a \geq A(x).$$

Conversely, if $\gamma(x,a) \leq \bigwedge_{y\in Y} \mu(y, B(y))$, then $\gamma(x,a) \leq \mu(y, B(y))$ for each $y \in Y$, i.e. $a \otimes B(y) \leq I(x,y)$, which yields $a \leq B(y) \to I(x,y)$, for each $y \in Y$, and hence $a \leq \bigwedge_{y\in Y} B(y) \to I(x,y) = A(x)$ which implies $\bigvee_{\gamma(x,a)\leq \bigwedge_{y\in Y} \mu(y,B(y))} a \leq A(x)$. We have proved $\varphi \circ \psi(A,B) = \langle A, B\rangle$.

It now suffices to show that ψ is monotone. If $u \leq v$ then for $\psi(u) = \langle A_u, B_u \rangle$ and $\psi(v) = \langle A_v, B_v \rangle$ we have $A_u(x) = \bigvee_{\gamma(x,a) \leq u} a$ and $A_v(x) = \bigvee_{\gamma(x,a) \leq v} a$ which implies $A_u(x) \leq A_v(x)$ for each $x \in X$, i.e. $\psi(u) \leq \psi(v)$. The proof of Theorem 5 is complete. □

Remark In [13], the author provides a natural interpretation of the fixed points and their lattices which is based on the approach to concepts as developed by traditional (Port-Royal) logic [1]. Fixed points are in [13] called formal concepts and the lattices of fixed points are called concept lattices. The theory of concept lattices thus established has been then developed in a series of papers. The intention is to have a theory of formal concepts with applications in concept data analysis and conceptual knowledge representation. Note that the application of lattices to data analysis have been suggested already by Birkhoff. Formal concepts and concept lattices are, in fact, models of sharp concepts and conceptual structures. The generalization to many-valued (fuzzy) case makes it possible to model also non-sharp concepts. We briefly outline the approach.

Let X represent the set of objects, Y the set of attributes, and let $I(g, m)$ be interpreted as the truth value of "the object x has the attribute y". Apparently, the fuzzy approach to I is appropriate since the relationship "to have" between objects and attributes is non-sharp. Call $\langle X, Y, I \rangle$ an $\mathbf{L}$-*context*. By traditional logic, a concept consists of a collection A (extent of concept) of objects and a collection B (intent of concept) of attributes such that (a) B consists of all attributes shared by all objects of A and (b) A consists of all objects sharing all the attributes of B. In the case of empiric objects, both extent and intent are non-sharp. Therefore, it seems reasonable to model A and B by $\mathbf{L}$-sets. Obviously, (a) and (b) are expressed by (5) and (6), respectively. It is thus natural to call a pair $\langle A, B \rangle \in L^X \times L^Y$ an $\mathbf{L}$-*concept* in $\langle X, Y, I \rangle$ iff $A^{\uparrow} = B$ and $B^{\downarrow} = A$, i.e. iff $\langle A, B \rangle$ is a fixed point. $\mathcal{L}(X, Y, I)$ is then called an $\mathbf{L}$-*concept lattice*. The partial order $\leq$ models the conceptual hierarchy on $\mathcal{L}(X, Y, I)$. The main role of $\mathcal{L}(X, Y, I)$ is to reveal the conceptual hierarchy hidden in the "input data" $\langle X, Y, I \rangle$.

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