LOGICAL PRECISION IN CONCEPT LATTICES

RADIM BĚLOHLÁVEK
Institute for Research and Applications of Fuzzy Modeling, University of Ostrava, Bráfova 7, 701 03 Ostrava, Czech Republic, belohlav@osu.cz
and
Department of Computer Science, Technical University of Ostrava, tř. 17. listopadu, 708 33 Ostrava-Poruba, Czech Republic

Abstract. Presented are new results on fuzzy concepts and concept lattices. Studied here is the issue of logical precision, a phenomenon that—although completely degenerate in two-valued logic—naturally arises in the context of fuzzy logic. We present basic results regarding logical precision and concept lattices. Since, as far as we know, the phenomenon has not been discussed in literature so far, we briefly discuss its role in general context of relational systems. We argue that logical precision should be considered as a possible way to treat the problem of information granularity from the logical point of view.

Key words: conceptual structure, logical precision, fuzzy concept lattice, granularity

1 INTRODUCTION

Human thinking is often identified with reasoning with concepts. On the intuitive level, formation of concepts (like MAMMAL, HIGH TEMPERATURE) are typical examples of what is meant by *information granulation* [11]: concepts can be thought of as granules of information which (on a higher level of abstraction) can be taken as units for reasoning. A formal theory of concepts and conceptual structures (so-called concept lattices) has been initiated in [10]. The theory of concept lattices is nowadays a well-elaborated one with applications in conceptual data analysis (see e.g. the survey in [7]). A generalization of concept lattices and related structures (like Galois connections and closure operators) from the point of view of fuzzy logic has been started in [2, 3]. Since the beginning, attention has been paid to phenomena which are hidden in the classical (i.e. bivalent) case, see e.g. [4]. The aim of this paper is to study another issue hidden in the classical case—so-called logical precision. In Section 2 we recall the fundamental notions and results. In Section 3 we study logical precision in concept lattices, Section 4 contains an illustrative example and conclusions.

2 FUZZY CONCEPTS AND FUZZY CONCEPT LATTICES

Following Port-Royal logic [1] (sometimes called traditional logic) a *concept* is determined by its *extent*, i.e. a collection of objects which fall under this concept, and its *intent*, i.e. a collection of properties (attributes) covered by this concept. Thus, e.g., the extent of the concept DOG consists of all dogs (or living dogs at a fixed time at a fixed world to avoid the philosophical problems) while the intent of DOG consists of all attributes of all dogs (i.e. "to bark", "to be a mammal", "to have four extremities" etc.). The primary relation appearing on concepts is that of hierarchical ordering. For instance, the concepts DOG and CAT are not comparable w.r.t. to conceptual hierarchy, while the concept MAMMAL is more abstract (greater in the hierarchy ordering) than both DOG and CAT. Started from these assumptions, Wille and his collaborators [10, 7] developed theory of so-called concept lattices with emphasis on (conceptual) data analysis. This theory is, in fact, a theory of sharp (bivalent) concepts: a given object either (fully) belongs to the extent of some concept or does (fully) not. The same holds for attributes. The sharpness of concepts is unrealistic from the point of view of fuzzy approach. Extents and intents of empiric concepts are not bivalent, they are vague. For instance, the extent of the concept HARD MINERAL, i.e. the collection of all hard minerals is certainly not bivalent.

Based on these motivations, the author started [2, 3, 4] the study of fuzzy concept lattices, i.e. Port-Royal concepts and concept lattices from the point of view of fuzzy logic. We now recall basic notions and results. Take any *complete residuated lattice* $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ as the structure of truth values. Residuated lattices play a prominent role as the algebraic structures of fuzzy logic in narrow sense [8, 9]. Recall that $\mathbf{L}$ is a complete residuated lattice if (1) $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice (with the least element 0, greatest element 1), (2) $\langle L, \otimes, 1\rangle$ is a commutative monoid, (3) $\otimes, \rightarrow$ are binary operations (called multiplication and residuum, respectively) which form an adjoint pair, i.e. $x \otimes y \leq z$ iff $x \leq y \rightarrow z$. The most studied and applied set of truth values is the real interval $[0, 1]$; three most important structures are given by Łukasiewicz, Gödel, and product multiplications (for details see e.g. [8]). Another important set of truth values is the set $\{a_0 = 0, a_1, \ldots, a_n = 1\}$ $(a_0 < \cdots < a_n)$ with $\otimes$ given by $a_k \otimes a_l = a_{\min(k,l)}$

and the corresponding $\to$ given by $a_k \to a_l = a_n$ for $a_k \leq a_l$ and $a_k \to a_l = a_l$ otherwise (this is a restriction of Gödel structure on $[0, 1]$). A complete first-order logic with semantics defined over complete residuated lattices can be found in [9]. For logical calculi defined over special classes of residuated lattices see [8]. In what follows, $\mathbf{L}$ (possibly with indices) always refers to a complete residuated lattice.

The primary notion is that of a *fuzzy context* ($\mathbf{L}$-context): it is a triple $\langle G, M, I\rangle$, where G and M are (crisp) sets interpreted as set of objects (G) and a set of properties (M) to which we restrict our attention, and $I \in L^{G\times M}$ is a fuzzy relation. The value $I(g, m) \in L$ is the truth value of the fact "the object $g \in G$ has the property $m \in M$". In accordance to the above sketched assumptions, *a (formal) fuzzy concept* (L-concept) is a pair $\langle A, B\rangle$, $A \in L^G$, $B \in L^M$, A plays the role of (vague) extent (fuzzy set of objects which determine the concept), B plays the role of (vague) intent (fuzzy set of properties which determine the concept), such that: (a) each object of A has all the properties of the intent B and (b) each property of B is shared by all the objects of the extent A. There are therefore two fundamental operators: $^{\uparrow}$ (and $^{\downarrow}$) which assign to each fuzzy set $A \in L^G$ ($B \in L^M$) of objects (of properties) the fuzzy set $A^{\uparrow} \in L^M$ ($B^{\downarrow} \in L^G$) of all the properties which are common to all objects of A (of all the objects which share all properties of B). Rewriting these linguistic descriptions on the semantical level of fuzzy logic we get formal definitions

$$A^{\uparrow}(m) = \bigwedge_{g\in G} (A(g) \to I(g, m)) \tag{1}$$

and

$$B^{\downarrow}(g) = \bigwedge_{m\in M} (B(m) \to I(g, m)) \tag{2}$$

which play the fundamental role. The condition for $\langle A, B\rangle$ to be a fuzzy concept is $A^{\uparrow} = B$ and $B^{\downarrow} = A$. Denote $\mathcal{B}(G, M, I) = \{\langle A, B\rangle \mid A^{\uparrow} = B, B^{\downarrow} = A\}$ the set of all fuzzy concepts given by the fuzzy context $\langle G, M, I\rangle$. The conceptual hierarchy is modeled by the relation $\leq$ defined on $\mathcal{B}(G, M, I)$ by

$$\langle A_1, B_1\rangle \leq \langle A_1, B_1\rangle \text{ iff } A_1 \subseteq A_2 (\text{iff } B_1 \supseteq B_2). \tag{3}$$

The following theorem generalizes the so-called main theorem of conceptual data analysis [3].

Theorem 1 *The set $\mathcal{B}(G, M, I)$ is under $\leq$ a complete lattice where the suprema and infima are given by*

$$\bigwedge_{j \in J} \langle A_j, B_j \rangle = \langle \bigcap_{j \in J} A_j, (\bigcup_{j \in J} B_j)^{\downarrow\uparrow} \rangle , \quad (4)$$

$$\bigvee_{j \in J} \langle A_j, B_j \rangle = \langle (\bigcup_{j \in J} A_j)^{\uparrow\downarrow}, \bigcap_{j \in J} B_j \rangle . \quad (5)$$

Moreover, an arbitrary complete lattice $\mathbf{V} = \langle V, \wedge, \vee \rangle$ is isomorphic to some $\mathcal{B}(G, M, I)$ iff there are mappings $\gamma : G \times L \to V$, $\mu : M \times L \to V$ such that $\gamma(G, L)$ is $\bigwedge$-dense in V, $\mu(M, L)$ is $\bigvee$-dense in V; $\alpha \otimes \beta \leq I(g, m)$ iff $\gamma(g, \alpha) \leq \mu(m, \beta)$.

The lattice $\mathcal{B}(G, M, I)$ is called a *fuzzy concept lattice* (**L**-concept lattice). Note that the lattice structure of concepts is very natural: to each set of concepts there is a concept which is the direct generalization (supremum) and a concept which is the direct specialization (infimum) of the concepts. Note also that in this perspective, the original (crisp) concepts lattices can be viewed as **L**-concept lattices where $L = \{0, 1\}$ is the set of truth values of classical logic. Given a context, the concept lattice reveals the conceptual structure hidden in the context (hidden dependencies of attributes, a guide for classification etc.) [10, 7, 3]. If the information in the context is fuzzy, the fuzzy conceptual data analysis is an appropriate tool.

3 LOGICAL PRECISION AND CONCEPT LATTICES

Consider a structure **L** of truth values. The set L is the set of all possible truth values which we have at disposal for logical modeling of our knowledge. It could be considered as representing "logical discernibility". Consider e.g. the two-element Boolean algebra. Then the level of discernibility is low—we can discern only fully true statements from fully false statements. An n-element chain of truth values offers more—we can discern n truth degrees. Very loosely, using more truth values means more *logical precision*.

From the point of view of information granularity [11], a fixed **L** represents a space of possible information granules (**L**-sets) in a fixed universe. It seems to be natural to have a possibility to change the set of truth values (in order to increase or decrease the logical discernibility) so that the structural properties of the model remain preserved. This task is successfully performed by humans: people can easily switch between rough reasoning and more precise reasoning by "focusing". In the following we propose a natural algebraic approach to the (change of) logical precision.

We are given a structure $\mathbf{L}_1$ of truth values using which our knowledge is formulated. For several reasons it might not be desirable to discern all the truth values of L_1 (e.g. for computational reasons, or, close truth values may appear essentially the same). In that case, a kind of rounding off is desirable. In order that the rounding off be systematic, it should be compatible with operations in $\mathbf{L}_1$. Formally, we look for another structure $\mathbf{L}_2$ of truth values such that a suitable onto morphism $h : L_1 \to L_2$ exists. In our case we require that h is an onto $\bigwedge$-morphism, i.e. h is an onto mapping such that $h(\bigwedge_{j\in J} a_j) = \bigwedge_{j\in J} h(a_j)$, $h(a \vee b) = h(a) \vee h(b)$, $h(a \otimes b) = h(a) \otimes h(b)$, $h(a \to b) = h(a) \to h(b)$, $h(0) = 0$, $h(1) = 1$ (following common usage, we use the same symbols for the corresponding operations in $\mathbf{L}_1$ and $\mathbf{L}_2$). The change-over from $\mathbf{L}_1$ to $\mathbf{L}_2$ then represents a decrease of logical precision: all values $a \in L_1$ rounded to the same element are identified, i.e. considered to be the same.

Let $\mathbf{L}_1$ and $\mathbf{L}_2$ be complete residuated lattices. Any mapping $h : L_1 \to L_2$ induces a mapping $h : L_1^X \to L_2^X$ by $(h(A))(x) = h(A(x))$ (we denote both of the mappings by the same symbol).

Intuitively, changing the structure of truth values in a systematic way should result in a systematic change of the structure of information (conceptual structure in our case). The following two statements confirm the intuition. Note that a lattice homomorphism $h : \mathbf{V}_1 \to \mathbf{V}_2$ between two complete lattices $\mathbf{V}_1$ and $\mathbf{V}_2$ is called *complete* if for each $K \subseteq V_1$ it holds $h(\bigwedge_{k\in K}) = \bigwedge_{k\in K} h(k)$ and $h(\bigvee_{k\in K}) = \bigvee_{k\in K} h(k)$.

Lemma 2 *Let $\mathbf{L}_1$, $\mathbf{L}_2$ be two complete residuated lattices and $h : \mathbf{L}_1 \to \mathbf{L}_2$ be an onto $\bigwedge$-homomorphism. Let $\langle G, M, I\rangle$ be an $\mathbf{L}_1$-context. Then for $C \in L_2^G$, $D \in L_2^M$, the following holds: $\langle C, D\rangle \in \mathcal{B}(G, M, h(I))$ iff there are $A \in L_1^G$, $B \in L_1^M$ such that $\langle A, B\rangle \in \mathcal{B}(G, M, I)$, $h(A) = C$, and $h(B) = D$.*

Proof. Let $\langle A, B\rangle$ be a concept in $\mathcal{B}(G, M, I)$, i.e. $A^\uparrow = B$, $B^\downarrow = A$. Due to the $\bigwedge$-preserving of h we have

$$\begin{aligned}
(h(A))^\uparrow(m) &= \bigwedge\{h(A)(g) \to (h(I))(g, m);\ g \in G\} = \\
&= \bigwedge\{h(A(g)) \to h(I(g, m));\ g \in G\} = \\
&= \bigwedge\{h(A(g) \to I(g, m));\ g \in G\} = \\
&= h(\bigwedge\{A(g) \to I(g, m);\ g \in G\}) = \\
&= h(A^\uparrow(m)) = (h(B))(m)\ ,
\end{aligned}$$

i.e. $h(A)^\uparrow = h(B)$. The proof of $h(B)^\downarrow = h(A)$ is symmetric. We have proved $\langle h(A), h(B)\rangle \in \mathcal{B}(G, M, I)$.

Conversely, let $\langle C, D\rangle$ be a concept in $\mathcal{B}(G, M, h(I))$. Take arbitrary $A \in L_1^G$ such that

$$h(A(g)) = C(g) \quad \text{for each } g \in G\ .$$

Such an A always exists due to the surjectivity of h. We have

$$\begin{aligned}
(h(A^{\uparrow})(m)) &= h(\bigwedge\{A(g) \to I(g,m);\ g \in G\}) = \\
&= \bigwedge\{h(A(g)) \to (h(I))(g,m);\ g \in G\} = \\
&= \bigwedge\{C(g) \to (h(I))(g,m);\ g \in G\} = \\
&= C^{\uparrow}(m) = D(m)\ ,
\end{aligned}$$

i.e., $h(A^{\uparrow}) = D$. Furthermore,

$$\begin{aligned}
(h(A^{\uparrow\downarrow})(g)) &= h(\bigwedge\{A^{\uparrow}(m) \to I(g,m);\ g \in G\}) = \\
&= \bigwedge\{h(A^{\uparrow}(m)) \to (h(I))(g,m);\ g \in G\} = \\
&= \bigwedge\{D(m) \to (h(I))(g,m);\ g \in G\} = \\
&= C^{\uparrow}(m) = D^{\downarrow}(g) = C(g)\ ,
\end{aligned}$$

i.e., $h(A^{\uparrow\downarrow}) = C$, and $\langle A^{\uparrow\downarrow}, A^{\uparrow}\rangle$ is the required concept from $\mathcal{B}(G, M, I)$. $\square$

Theorem 3 *Under the conditions of the preceding lemma, there is a complete homomorphism of* $\mathcal{B}(G, M, I)$ *onto* $\mathcal{B}(G, M, h(I))$.

Proof. Due to Lemma 2 (and its proof) it suffices to show that the mapping $h^* : \mathcal{B}(G, M, I) \to \mathcal{B}(G, M, h(I))$ defined by

$$h^*(A, B) = \langle h(A), h(B)\rangle$$

preserves arbitrary meets and joins. We proceed only for meets, the case of joins may be proved analogously. Let $\langle A_j, B_j\rangle \in \mathcal{B}(G, M, I)$, $j \in J$. We have to prove

$$h^*(\bigwedge_{j\in J} \langle A_j, B_j\rangle) = \bigwedge_{j\in J} h^*(\langle A_1, B_1\rangle). \tag{6}$$

By Theorem 1 we have

$$\bigwedge_{j\in J} \langle A_j, B_j\rangle = \langle \bigcap_{j\in J} A_j, (\bigcap_{j\in J} A_j)^{\uparrow}\rangle. \tag{7}$$

We thus conclude

$$\begin{aligned} & h^*(\bigwedge_{j\in J}\langle A_j, B_j\rangle) = \\ = \; & h^*(\langle\bigcap_{j\in J} A_j, (\bigcap_{j\in J} A_j)^{\uparrow}\rangle) = \langle h(\bigcap_{j\in J} A_j), h((\bigcap_{j\in J} A_j)^{\uparrow})\rangle \end{aligned} \tag{8}$$

and

$$\begin{aligned} & \bigwedge_{j\in J} h^*(\langle A_j, B_j\rangle) = \\ = \; & \bigwedge_{j\in J}\langle h(A_j), h(B_j)\rangle = \langle\bigcap_{j\in J} h(A_j), (\bigcap_{j\in J} h(A_j))^{\uparrow}\rangle. \end{aligned} \tag{9}$$

The concepts in (8) and (9) equal iff their extents equal. Consider thus any $g \in G$. For the extents of the concepts in (8) and (9) we have

$$\begin{aligned} & (h(\bigcap_{j\in J} A_j))(g) = h((\bigcap_{j\in J} A_j)(g)) = \\ = \; & \bigwedge_{j\in J} h(A_j(g)) = \bigwedge_{j\in J}(h(A_j))(g) = (\bigcap_{j\in J} h(A_j))(g), \end{aligned}$$

i.e. the concepts in (8) and (9) are the same. We have proved (6). By the above considerations, the proof is finished. □

Remark 1 The foregoing theorem is relevant also from the application point of view. Suppose we have a concept lattice with truth values from L_1. A further analysis on the level of L_1 may be (from various reasons, e.g. computational ones) "too precise". We can then skip to the level of $L_2 = h(L_1)$ which is appropriate. Due to Theorem 3, the structure of concepts changes systematically, i.e. the structure of concepts in L_1 is in a systematic way more precise than that one in L_2. An illustrating example is presented at the end of this paper.

Consider now another application of logical precision. The structure $\mathbf{L}$ of truth values in which our knowledge is formulated may be in some way composed of structures $\mathbf{L}_i$ ($i \in I$). Since $\mathbf{L}_i$ are more simple, it is to be expected that if possible, the analysis and processing of knowledge formulated using $\mathbf{L}$ can be simplified by a suitable decomposition of the knowledge into parts which correspond to particular $\mathbf{L}_i$'s. In the following, we concentrate on the case where "$\mathbf{L}$ is composed of $\mathbf{L}_i$'s" means that $\mathbf{L}$ is a (sub)direct product of $\mathbf{L}_i$ ($i \in I$).

First, recall that a residuated lattice $\mathbf{L}$ is a subdirect product of residuated lattices $\mathbf{L}_i$ $(i \in I)$ if $\mathbf{L}$ is a complete subalgebra of the direct product $\times_{i \in I} \mathbf{L}_i$ such that $\mathrm{pr}_i(L) = L_i$ (where $\mathrm{pr}_i : \times_{i \in I} L_i \to L_i$ is the i-th projection defined for $a = \langle \ldots, a_j, \ldots \rangle \in \times_{i \in I} L_i$ by $\mathrm{pr}_i(a) = a_i$). If $\mathbf{L}_1, \mathbf{L}_2$ are complete residuated lattices, we say that $\mathbf{L}_1$ is a complete subalgebra of $\mathbf{L}_1$ if it is a subalgebra such that arbitrary infima and suprema in $\mathbf{L}_1$ coincide with those in $\mathbf{L}_2$. If $\mathbf{L}$ is a subdirect product of complete residuated lattices $\mathbf{L}_i$ $(i \in I)$ such that $\mathbf{L}$ is, moreover, a complete subalgebra of $\times_{i \in I} \mathbf{L}_i$ we say that $\mathbf{L}$ is a *complete subdirect product of* $\mathbf{L}_i$ *$(i \in I)$* . Clearly, the direct product is a special case of a subdirect product.

Remark 2 Note that the above condition is not trivial: Let $\mathbf{L}_1 = \mathbf{L}_2$ be complete residuated lattices given on $[0,1]$ by the Gödel structure (i.e. $\otimes = \min$), let $\mathbf{L}$ be the subalgebra of $\mathbf{L}_1 \times \mathbf{L}_2$ given by $L = (\{0\} \cup (0.5, 1]) \times (\{0\} \cup (0.5, 1]) \cup (\{1\} \cup [0, 0.5]) \times (\{1\} \cup [0, 0.5])$. Then $\mathbf{L}$ is a subdirect product of $\mathbf{L}_1$ and $\mathbf{L}_2$ which a complete residuated lattice but not a complete subalgebra of $\mathbf{L}_1 \times \mathbf{L}_2$. Indeed, the infimum of $\{\langle a, 0.6 \rangle \mid 0.5 < a \leq 1\}$ is $\langle 0, 0.6 \rangle$ in $\mathbf{L}$; the infimum of $\{\langle a, 0.6 \rangle \mid 0.5 < a \leq 1\}$ is $\langle 0.5, 0.6 \rangle$ in $\mathbf{L}_1 \times \mathbf{L}_2$.

Remark 3 Let $\mathbf{L}_i$ $(i \in I)$ be complete residuated lattices. One easily verifies the following propositions.

(1) The direct product $\times_{i \in I} \mathbf{L}_i$ is a complete residuated lattice.

(2) It $\mathbf{L}$ is a complete subdirect product of $\mathbf{L}_i$ $(i \in I)$ then $\mathrm{pr}_i : L \to L_i$ is a complete morphism from $\mathbf{L}$ onto $\mathbf{L}_i$ for any $i \in I$.

Let $\mathbf{L}_1, \mathbf{L}_2$ be complete residuated lattices such that $\mathbf{L}_1$ is a complete subalgebra of $\mathbf{L}_2$. If $\langle G, M, I \rangle$ is an $\mathbf{L}_2$-context such that $I(g, m) \in L_1$ for all $g \in G$ and $m \in M$ then $\langle G, M, I \rangle$ may be considered as an $\mathbf{L}_1$-context as well. In that case, we denote by $\mathcal{B}_{\mathbf{L}_1}(G, M, I)$ and $\mathcal{B}_{\mathbf{L}_2}(G, M, I)$ the $\mathbf{L}_1$-concept lattice and the $\mathbf{L}_2$-concept lattice, respectively, corresponding to $\langle G, M, I \rangle$.

Lemma 4 *If $\mathbf{L}_1$ and $\mathbf{L}_2$ are complete residuated lattices such that $\mathbf{L}_1$ is a complete subalgebra of $\mathbf{L}_2$ and $\langle G, M, I \rangle$ is an $\mathbf{L}_2$-context such that $I(g, m) \in L_1$ for any $g \in G$ and $m \in M$, then $\mathcal{B}_{\mathbf{L}_1}(G, M, I)$ is a complete sublattice of $\mathcal{B}_{\mathbf{L}_2}(G, M, I)$.*

Proof. Let $A \in L_1^G$. Denote by ${}^{\uparrow_1}$ and ${}^{\uparrow_2}$ the operators defined by I using the operations of $\mathbf{L}_1$ and $\mathbf{L}_2$, respectively. Since $\mathbf{L}_1$ is a complete residuated sublattice of $\mathbf{L}_2$, a moment reflection shows that $A^{\uparrow_1} = A^{\uparrow_2}$; similarly for ${}^{\downarrow_1}$ and ${}^{\downarrow_2}$. We immediately conclude $\mathcal{B}_{\mathbf{L}_1}(G, M, I) \subseteq \mathcal{B}_{\mathbf{L}_2}(G, M, I)$. Moreover, from (4) and (5) we infer that $\mathcal{B}_{\mathbf{L}_1}(G, M, I)$ is a complete sublattice of $\mathcal{B}_{\mathbf{L}_2}(G, M, I)$. □

Let $\mathbf{L}$ be a subalgebra of a direct product $\times_{j\in J}\mathbf{L}_j$ of complete residuated lattices. For an $\mathbf{L}$-relation I between G and M, and each $j \in J$ we define an $\mathbf{L}_j$-relation between G and M by

$$I_j(g, m) = \mathrm{pr}_j(I(g, m)).$$

Let, moreover, $\mathbf{L} = \times_{j\in J}\mathbf{L}_j$. For $\mathbf{L}_j$-relations I_j between G and M we define an $\mathbf{L}$-relation $\times_{j\in J}I_j$ between G and M by

$$\mathrm{pr}_j((\times_{i\in J}I_i)(g, m)) = I_j(g, m)$$

for any $g \in G$, $m \in M$, $k \in J$. Thus, for instance, if $J = \{1, 2\}$ then $I_1 \times I_2(g, m) = \langle I_1(g, m), I_2(g, m)\rangle$. Furthermore, for $\mathbf{L}_j$ contexts $\langle G, M, I_j\rangle$ $(j \in J)$ we put $\times_{j\in J}\langle G, M, I_j\rangle = \langle G, M, \times_{j\in J}I_j\rangle$. We have the following theorem (note that $\cong$ denotes "is isomorphic to").

Theorem 5 *Let $\langle G, M, I_j\rangle$ be an $\mathbf{L}_j$-context for any $j \in J$. Then $\mathcal{B}(\times_{j\in J}\langle G, M, I_j\rangle) \cong \times_{j\in J}\mathcal{B}(G, M, I_j)$.*

Proof. We prove the theorem by showing that there is an isotone bijection $\varphi : \mathcal{B}(\times_{j\in J}\langle G, M, I_j\rangle) \to \times_{j\in J}\mathcal{B}(G, M, I_j)$. For $\langle A, B\rangle \in \mathcal{B}(\times_{j\in J}\langle G, M, I_j\rangle)$ define $\varphi(A, B)$ by

$$\mathrm{pr}_j(\varphi(A, B)) = \langle A_j, B_j\rangle$$

where $A_j(g) = \mathrm{pr}_j(A(g))$, $B_j(g) = \mathrm{pr}_j(B(g))$. By Remark 3, $pr_j : \times_{i\in J}\mathbf{L}_i \to \mathbf{L}_j$ $(j \in J)$ are complete onto morphisms. Furthermore, by the definition of $\times_{j\in J}I_j$ we have $\mathrm{pr}_j((\times_{i\in J}I_i)(g, m)) = I_j(g, m)$. Therefore, by Theorem 3, the mappings $\varphi_j : \langle A, B\rangle \mapsto \langle A_j, B_j\rangle$ $(j \in J)$, defined by $A_j(g) = \mathrm{pr}_j(A(g))$, $B_j(m) = \mathrm{pr}_j(B(m))$, are complete onto morphisms of $\mathcal{B}(\times_{j\in J}\langle G, M, I_j\rangle)$ onto $\mathcal{B}(G, M, I_j)$. This shows that φ is correctly defined and, moreover, is isotone. The injectivity of φ is immediate from the definition. It remains to show that φ is an onto mapping. Take therefore $\langle A_j, B_j\rangle \in \mathcal{B}(G, M, I_j)$ for each $j \in J$. Take $\langle A, B\rangle \in L^G \times L^M$ so that $\mathrm{pr}_j(A(g)) = A_j(g)$ and $\mathrm{pr}_j(B(m)) = B_j(m)$. Clearly, $\mathrm{pr}_j(\varphi(A, B)) = \langle A_j, B_j\rangle$. We need to show that $\langle A, B\rangle \in \mathcal{B}(\times_{j\in J}\langle G, M, I_j\rangle)$, i.e. we need to verify $A^{\uparrow} = B$ and $B^{\downarrow} = A$. Due to the similarity of both of the cases we proceed for the first case only.

Consider any $j \in J$. Since $\mathrm{pr}_j : \times_{i \in J} \mathbf{L}_i \to \mathbf{L}_j$ is a complete morphism, we have

$$\begin{aligned} & \mathrm{pr}_j(A^{\uparrow}(m)) = \mathrm{pr}_j(\bigwedge_{g \in G} A(g) \to I(g,m)) = \\ = \quad & \bigwedge_{g \in G} \mathrm{pr}_j(A(g)) \to \mathrm{pr}_j(I(g,m)) = \bigwedge_{g \in G} A_j(g) \to I_j(g,m) = \\ = \quad & B_j(m) = \mathrm{pr}_j(B(m)) \end{aligned}$$

i.e. $A^{\uparrow} = B$ is true. The proof is complete. $\square$

As a combination of the preceding results we get the following assertion.

Theorem 6 *Let $\langle G, M, I \rangle$ be an $\mathbf{L}$-context. If $\mathbf{L}$ is a complete subdirect product of $\mathbf{L}_j$ $(j \in J)$, then the $\mathbf{L}$-concept lattice $\mathcal{B}(G, M, I)$ is isomorphic to a subdirect product of $\mathbf{L}_j$-concept lattices $\mathcal{B}(G, M, \mathrm{pr}_j(I))$. Each $\mathcal{B}(G, M, \mathrm{pr}_j(I))$ is a complete homomorphic image of $\mathcal{B}(G, M, I)$.*

Proof. Since $I = \times_{j \in J} \mathrm{pr}_j(I)$, Theorem 5 yields the existence of an isomorphism φ from $\mathcal{B}_{\times_{j \in J} \mathbf{L}_j}(G, M, I)$ to $\times_{j \in J} \mathcal{B}_{\mathbf{L}_j}(G, M, \mathrm{pr}_j(I))$ (in what follows, φ refers to the isomorphism constructed in the proof of Theorem 5). Furthermore, by Lemma 4, $\mathcal{B}_{\mathbf{L}}(G, M, I)$ is a complete sublattice of $\mathcal{B}_{\times_{j \in J} \mathbf{L}_j}(G, M, I)$. Therefore, it is sufficient to prove that for each $j \in J$ and each $\langle A_j, B_j \rangle \in \mathcal{B}_{\mathbf{L}_j}(G, M, \mathrm{pr}_j(I))$ there is an $\langle A, B \rangle \in \mathcal{B}_{\mathbf{L}}(G, M, I)$ such that $\mathrm{pr}_j(\varphi(A, B)) = \langle A_j, B_j \rangle$. Take any $\langle A_j, B_j \rangle \in \mathcal{B}_{\mathbf{L}_j}(G, M, \mathrm{pr}_j(I))$. Consider any $g \in G$. As $\mathbf{L}$ is a subdirect product of $\mathbf{L}_j$'s, there exists $a_g \in L$ such that $\mathrm{pr}_j(a_g) = A_j(g)$. Now, take $A' \in L^G$ such that $A'(g) = a_g$, and put $B = A'^{\uparrow}$, $A = B^{\downarrow}$. We have $\langle A, B \rangle \in \mathcal{B}_{\mathbf{L}}(G, M, I)$. A moment inspection shows that $\mathrm{pr}_j(A(g)) = A_j(g)$ and $\mathrm{pr}_j(B(m)) = B_j(m)$ for any $g \in G$ and $m \in M$, i.e. $\mathrm{pr}_j(\varphi(A, B)) = \langle A_j, B_j \rangle$ completing the proof. $\square$.

Theorem 6 is especially interesting in view of the following well-known fact due to Birkhoff [6]: every algebra is a subdirect product of subdirectly irreducible algebras. A moment reflection shows that the method can be modified in a straightforward way to obtain: Each complete residuated lattice is a subdirect product of subdirectly irreducible complete residuated lattices. In a sense, subdirectly irreducible residuated lattices can be thought of as elementary structures of truth values of which any structure of truth values is composed. Theorem 6 can thus be rephrased as: each conceptual structure is composed of conceptual structures built over elementary structures of truth values. Another feature of the structure of truth values, often

I	size		from sun	
	small (ss)	large (sl)	far (df)	near (dn)
Mercury (Me)	1	0	0	1
Venus (V)	1	0	0	1
Earth (E)	1	0	0	1
Mars (Ma)	1	0	$\frac{1}{2}$	1
Jupiter (J)	0	1	1	$\frac{1}{2}$
Saturn (S)	0	1	1	$\frac{1}{2}$
Uranus (U)	$\frac{1}{2}$	$\frac{1}{2}$	1	0
Neptune (N)	$\frac{1}{2}$	$\frac{1}{2}$	1	0
Pluto (P)	1	0	1	0

Table 1: $\mathbf{L}_1$-context $\langle G, M, I\rangle$.

emphasized esp. from the application point of view, is the linear ordering of truth values. A modification of [9, Theorem 4.8] yields that a complete residuated lattice $\mathbf{L}$ is a subdirect product of linearly ordered complete residuated lattices iff it satisfies prelinearity, i.e. $(x \to y) \vee (y \to x) = 1$ is valid in $\mathbf{L}$. We therefore have the following corollary.

Corollary 7 *Each $\mathbf{L}$-concept lattice is a subdirect product of $\mathbf{L}_j$-concept lattices such that the structures of truth values $\mathbf{L}_j$ are subdirectly irreducible. If, moreover, $\mathbf{L}$ satisfies prelinearity, then each $\mathbf{L}_j$ is linearly ordered.*

4 EXAMPLE, CONCLUSIONS

Example. We present an example demonstrating Theorem 3. Put $L_1 = \{0, \frac{1}{2}, 1\}$, $L_2 = \{0, 1\}$, and let $\mathbf{L}_1$ and $\mathbf{L}_2$ be residuated lattices described in Section 2. Let G contain nine elements (Mercury,..., Pluto), and M contain four attributes ("size small",..., "near to sun"). Consider the homomorphism given by $h(0) = 0$, $h(\frac{1}{2}) = h(1) = 1$. The contexts $\langle G, M, I\rangle$ and $\langle G, M, h(I)\rangle$ are given in Table 1 and Table 2.

The elements of $\mathcal{B}(G, M, I)$ and $\mathcal{B}(G, M, h(I))$, i.e. $\mathbf{L}_1$-concepts and $\mathbf{L}_2$-concepts are given by Tab. 3. The fuzzy concept lattice $\mathcal{B}(G, M, I)$ is depicted in Fig. 1. By Theorem 3, $\mathcal{B}(G, M, h(I))$ is a homomorphic image of $\mathcal{B}(G, M, I)$. The homomorphism h^* is indicated in Fig. 2, e.g. $h^*(1) = 1$, $\ldots$, $h^*(25) = 12$. The fuzzy concept lattice $\mathcal{B}(G, M, h(I))$ is depicted in Fig. 3.

$h(I)$	size small (ss)	size large (sl)	from sun far (df)	from sun near (dn)
Mercury (Me)	1	0	0	1
Venus (V)	1	0	0	1
Earth (E)	1	0	0	1
Mars (Ma)	1	0	1	1
Jupiter (J)	0	1	1	1
Saturn (S)	0	1	1	1
Uranus (U)	1	1	1	0
Neptune (N)	1	1	1	0
Pluto (P)	1	0	1	0

Table 2: $\mathbf{L}_2$-context $\langle G, M, h(I) \rangle$.

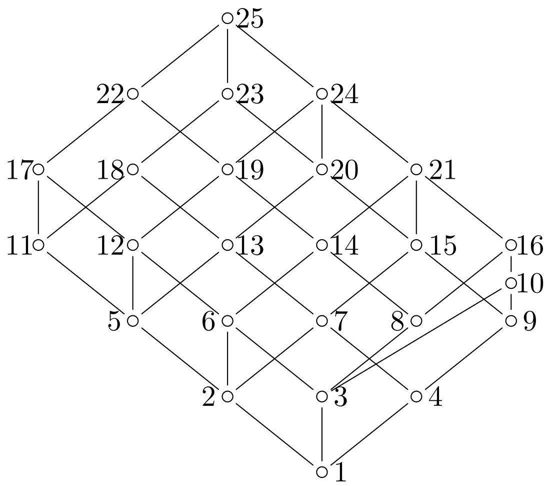

Figure 1: Concept lattice of the context in Tab. 1.

no.	extent									intent			
	Me	V	E	Ma	J	S	U	N	P	ss	sl	df	dn
1.	0	0	0	0	0	0	0	0	0	1	1	1	1
2.	0	0	0	$\frac{1}{2}$	0	0	0	0	0	1	0	1	1
3.	0	0	0	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0	1	1	1	0
4.	0	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	0	1	1	1
5.	0	0	0	1	0	0	0	0	0	1	0	$\frac{1}{2}$	1
6.	0	0	0	$\frac{1}{2}$	0	0	$\frac{1}{2}$	$\frac{1}{2}$	1	1	0	1	0
7.	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	0	0	1	1
8.	0	0	0	0	0	0	1	1	0	$\frac{1}{2}$	$\frac{1}{2}$	1	0
9.	0	0	0	0	1	1	0	0	0	0	1	1	$\frac{1}{2}$
10.	0	0	0	0	1	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	1	1	0
11.	1	1	1	1	0	0	0	0	0	1	0	0	1
12.	0	0	0	1	0	0	$\frac{1}{2}$	$\frac{1}{2}$	1	1	0	$\frac{1}{2}$	0
13.	0	0	0	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	0	0	$\frac{1}{2}$	1
no.	extent									intent			
	Me	V	E	Ma	J	S	U	N	P	ss	sl	df	dn
14.	0	0	0	$\frac{1}{2}$	0	0	1	1	1	$\frac{1}{2}$	0	1	$\frac{1}{2}$
15.	0	0	0	$\frac{1}{2}$	1	1	0	0	0	0	0	1	$\frac{1}{2}$
16.	0	0	0	0	1	1	1	1	0	0	$\frac{1}{2}$	1	0
17.	1	1	1	1	0	0	$\frac{1}{2}$	$\frac{1}{2}$	1	1	0	0	0
18.	1	1	1	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	0	0	0	1
19.	0	0	0	1	0	0	1	1	1	$\frac{1}{2}$	0	$\frac{1}{2}$	0
20.	0	0	0	1	1	1	0	0	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$
21.	0	0	0	$\frac{1}{2}$	1	1	1	1	1	0	0	1	0
22.	1	1	1	1	0	0	1	1	1	$\frac{1}{2}$	0	0	0
23.	1	1	1	1	1	1	0	0	0	0	0	0	$\frac{1}{2}$
24.	0	0	0	1	1	1	1	1	1	0	0	$\frac{1}{2}$	0
25.	1	1	1	1	1	1	1	1	1	0	0	0	0

Table 3: Fuzzy concepts of the context of Tab. 1.

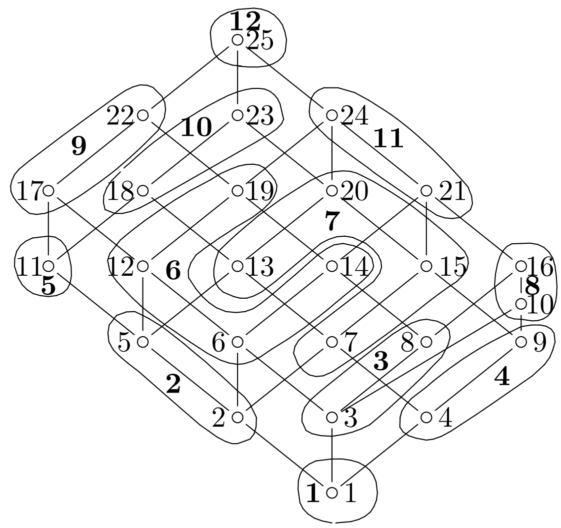

Figure 2: Classes of the congruence relation induced by h.

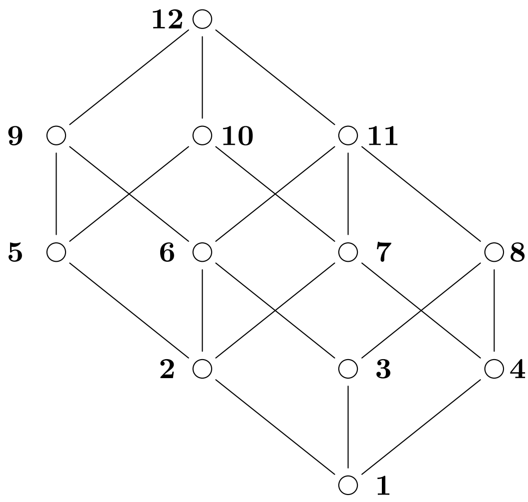

Figure 3: Lattice $\mathcal{B}(G, M, h(I))$.

Conclusion. We presented a method of simplification (factorization) of fuzzy concept lattice based on the concept of logical precision. Another method of factorization (based on the concept of similarity relation) was presented in [4]. Furthermore, we presented (sub)direct-like representation theorems for fuzzy concept lattices motivated by the concept of logical precision.

Both of the concepts, logical precision and similarity, are degenerate in bivalent logic and seem to play important role in fuzzy logic. General results from the point of view of first-order fuzzy logic are being prepared [5].

Further results on simplification and representation of fuzzy concept lattices are being prepared.

Acknowledgement Supported by grant no. 201/99/P060 of the GA ČR and by NATO Advanced Fellowship B for 2000. Part of the paper written during author's visit in the Center for Intelligent Systems, SUNY-Binghamton. The support by the Center and by its director, Professor G. J. Klir, is gratefully acknowledged.

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