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Lattice-type fuzzy order is uniquely given by its 1-cut: proof and consequences

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Abstract

A 1-cut of a fuzzy relation (sometimes called a core) does not contain all the information that is represented by the fuzzy relation. Particularly, a fuzzy order \leq on a universe X equipped with an fuzzy equality \approx is not uniquely determined by its 1-cut $^1 \leq = \{(x, y) \mid (x \leq y) = 1\}$. That is, there are in general several fuzzy orders with a common 1-cut. We show that, if the fuzzy order obeys in addition the lattice structure (which many natural examples of fuzzy orders do), it is uniquely determined by its 1-cut. Moreover, we discuss consequences of this result for the so-called fuzzy concept lattices and formal concept analysis.

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1. Introduction

Hierarchical structures are ubiquitous in everyday human affairs. That is why the partial orders are among the most important relations. From the point of view of fuzzy approach, the notion of a partial order gets its natural generalization to fuzzy setting. The so-called fuzzy orders were investigated in a number of papers (Zadeh's [16] being the first one; since then let me recall [6,7,10,13,5] where one can find further information). Although this has not been always clearly addressed, there are natural examples of fuzzy orders. The very reason for this is that partial order models the relationship of being subsumed (in a certain sense) and that the relationship of being subsumed is in general a

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1 fuzzy one. Thus the degree $(x \leq y)$ to which a fuzzy order \leq applies to x and y gets its natural
2 meaning as the degree to which y subsumes x .

3 It is well-known that given a fuzzy relation R , the a -cut does not in general uniquely determine
4 R . This is particularly true of 1-cuts: For a fuzzy relation R there may exist another fuzzy relation
5 R' such that ${}^1R = {}^1R'$ (their 1-cuts are equal). A natural question arises of whether there are some
6 natural conditions under which an a -cut or some collection of a -cuts of a given fuzzy relation
7 does uniquely determine the fuzzy relation. For example, a well-known representation theorem says
8 that each fuzzy relation R is uniquely given by the collection $\{{}^aR \mid a \in [0, 1]\}$ of all of its a -cuts.
9 Moreover, if the fuzzy relation is crisp then, obviously, it is uniquely given by its 1-cut (its “fully
10 true”-part). But these examples are in a sense rather trivial.

11 The aim of this paper is to show that in case of special types of fuzzy orders, 1-cuts contain
12 all the information contained in the fuzzy order. First, we show that general fuzzy orders are not
13 uniquely determined by their 1-cuts. Second, we show that if a fuzzy order \leq is of a lattice type (in
14 the usual sense that infima and suprema exist) then it is uniquely given by its 1-cut ${}^1\leq$. Also, using
15 the so-called fuzzy concept lattices, we show that lattice-type fuzzy orders form a natural class of
16 fuzzy orders with important examples including the set of all fuzzy sets in a given universe.

17 2. The result

18 First, we recall the necessary notions. We will use complete residuated lattices as structure of
19 truth values. Doing so, we attain a sufficient level of generality whereas the most commonly used
20 structures of truth values resulting from a given (left-continuous) t-norm on $[0, 1]$ become the most
21 important examples.

22 Complete residuated lattices, being introduced in the 1930s in ring theory, were introduced into
23 the context of fuzzy logic by Goguen [8,9]. Fundamental contribution to formal fuzzy logic using
24 residuated lattices as the structures of truth values is due to Pavelka [15]. Later on, various logical
25 calculi were investigated using residuated lattices or particular types of residuated lattices. A thorough
26 information about the role of residuated lattices in fuzzy logic and fuzzy set theory can be obtained
27 from [5,10,11,14]. In the following, \mathbf{L} denotes an arbitrary complete residuated lattice. Recall that
28 a (complete) residuated lattice is an algebra $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1 \rangle$ such that $\langle L, \wedge, \vee, 0, 1 \rangle$ is a
29 (complete) lattice with the least element 0 and the greatest element 1, $\langle L, \otimes, 1 \rangle$ is a commutative
30 monoid, and \otimes, \rightarrow form an adjoint pair, i.e. $a \otimes b \leq c$ iff $a \leq b \rightarrow c$ is valid for any $a, b, c \in L$.

31 **Remark 1.** (1) Given any left-continuous t-norm \otimes on $[0, 1]$ and putting $a \rightarrow b = \sup\{z \mid a \otimes z \leq b\}$,
32 $\langle [0, 1], \min, \max, \otimes, \rightarrow, 0, 1 \rangle$ is a complete residuated lattice. Conversely, for each complete residuated
33 lattice $\langle [0, 1], \min, \max, \otimes, \rightarrow, 0, 1 \rangle$, \otimes is a left-continuous t-norm (see e.g. [5]).

34 (2) Particularly, the three basic continuous t-norms and their corresponding residua are the
35 Łukasiewicz one ($a \otimes b = \max(a + b - 1, 0)$, $a \rightarrow b = \min(1 - a + b, 1)$), the Gödel one ($a \otimes b =$
36 $\min(a, b)$, $a \rightarrow b = 1$ if $a \leq b$ and $= b$ else), and the product one ($a \otimes b = a \cdot b$, $a \rightarrow b = 1$ if $a \leq b$ and
37 $= b/a$ else).

38 All properties of complete residuated lattices used in the sequel are well known and can be
39 found in any of the above-mentioned monographs. Note that particular types of residuated lattices

(distinguishable by identities) include Boolean algebras, Heyting algebras, algebras of Girard's linear logic, MV-algebras, Gödel algebras, product algebras, and more generally, BL-algebras (see [11,12]). An \mathbf{L} -set (or fuzzy set with truth degrees in \mathbf{L}) in a universe set U is any mapping $A: U \rightarrow L$, $A(u) \in L$ being interpreted as the truth value of “ u belongs to A ”. For $A_1, A_2: U \rightarrow L$ we put $A_1 \subseteq A_2$ iff $A_1(u) \leq A_2(u)$ for each $u \in U$. If $U = V \times V$, A is called a binary \mathbf{L} -relation on V . Recall that \mathbf{L} -equivalence (\mathbf{L} -similarity) on a set U is a binary \mathbf{L} -relation E on U satisfying $E(u, u) = 1$ (reflexivity), $E(u, v) = E(v, u)$ (symmetry), and $E(u, v) \otimes E(v, w) \leq E(u, w)$ (transitivity). An \mathbf{L} -equivalence on U for which $E(u, v) = 1$ implies $u = v$ will be called an \mathbf{L} -equality. A binary \mathbf{L} -relation R on U is compatible with a binary \mathbf{L} -relation E on U if for any $u_1, v_1, u_2, v_2 \in U$ we have $R(u_1, v_1) \otimes E(u_1, v_1) \otimes E(u_2, v_2) \leq R(u_2, v_2)$. If E is an \mathbf{L} -equivalence then compatibility means that if u_1 and v_1 are R -related and u_1 is equivalent to u_2 and v_1 is equivalent to v_2 then u_2 and v_2 are R -related as well.

Recall that a crisp order (sometimes called a partial order) is a binary relation of a set which is reflexive, antisymmetric, and transitive. We now present a definition of fuzzy order.

Definition 1 (Bělohlávek). An \mathbf{L} -order (or a fuzzy order) on a set X with an \mathbf{L} -equality relation \approx is a binary \mathbf{L} -relation \leq which is compatible w.r.t. \approx and satisfies

$$\begin{aligned} x &\leq x = 1, & (\text{reflexivity}), \\ (x &\leq y) \wedge (y &\leq x) &\leq x \approx y, & (\text{antisymmetry}), \\ (x &\leq y) \otimes (y &\leq z) &\leq x &\leq z, & (\text{transitivity}). \end{aligned}$$

If \leq is an \mathbf{L} -order on a set X with an \mathbf{L} -equality \approx , we call the pair $\mathbf{X} = \langle \langle X, \approx \rangle, \leq \rangle$ an \mathbf{L} -ordered set.

Remark 2. (1) Since we want to have an appropriate generalization of the axioms of partial order, and particularly of the antisymmetry axiom, an \mathbf{L} -equality on the universe X is needed.

(2) Zadeh defined a fuzzy order as a binary relation R which is reflexive and transitive (in the standard way) and satisfies that $x = y$ whenever $R(x, y) > 0$ and $R(y, x) > 0$ (Zadeh calls this property antisymmetry). It is easy to see that Zadeh's antisymmetry is equivalent to our antisymmetry when the \mathbf{L} -equality \approx is a crisp identity; thus, Zadeh's fuzzy order is a special case of our general concept of an \mathbf{L} -order.

(3) Clearly, if \mathbf{L} is the two-element Boolean algebra of bivalent logic, the notion of \mathbf{L} -order coincides with the usual notion of (partial) order.

(4) Note that antisymmetry is defined by requiring $R(x, y) \otimes R(y, x) \leq (x \approx y)$ in [10] (with \approx being the crisp equality), and [13,7] (\approx being a fuzzy equivalence). This is a different concept to which the results presented below do not apply. Note also that in both [10,7], the authors use $[0, 1]$ equipped with a (left-continuous) t-norm as the structure of truth values, while in [13], the authors use complete residuated lattices.

For an \mathbf{L} -order \leq on $\langle X, \approx \rangle$ denote by ${}^1\leq$ its 1-cut, i.e.

$${}^1\leq = \{ \langle x, y \rangle \mid (x \leq y) = 1 \}.$$

${}^1\leq$ can be seen as the “fully true”-part of \leq in that it contains those pairs $\langle x, y \rangle$ for which the fact “ x is under y ” has the truth value 1. It is well-known that in general, a fuzzy set (fuzzy relation)

Table 1
Fuzzy equality \approx and fuzzy order \preceq defined on $X = \{x_1, x_2, x_3, x_4\}$

\approx	x_1	x_2	x_3	x_4	\preceq	x_1	x_2	x_3	x_4
x_1	1	0.2	0.2	0.1	x_1	1	1	1	1
x_2	0.2	1	0.5	0.1	x_2	0.2	1	0.5	1
x_3	0.2	0.5	1	0.1	x_3	0.2	0.6	1	1
x_4	0.1	0.1	0.1	1	x_4	0.1	0.1	0.1	1

1 is not uniquely represented by its 1-cut. The following example illustrates that this is true in case of a fuzzy order as well.

3 **Example 1.** Let $L = [0, 1]$ be equipped with the Gödel structure (i.e. the t-norm \otimes being min).
 Consider a universe $X = \{x_1, x_2, x_3, x_4\}$, an \mathbf{L} -equality \approx given in Table 1 (left), and an \mathbf{L} -relation
 5 \preceq on X given in Table 1 (right). One can easily verify that \preceq is an \mathbf{L} -order on $\langle X, \approx \rangle$. If one defines
 an \mathbf{L} -relation \preceq' on X the same way as \preceq except for $\preceq'(x_3, x_2) = 0.5$ then \preceq' is an \mathbf{L} -order on $\langle X, \approx \rangle$
 7 again. Now, ${}^1\preceq = {}^1\preceq' = \text{id}_X \cup \{\langle x_1, x_2 \rangle, \langle x_1, x_3 \rangle, \langle x_1, x_4 \rangle, \langle x_2, x_4 \rangle, \langle x_3, x_4 \rangle\}$, i.e. the two different \mathbf{L} -orders agree on their 1-cut.

9 Example 1 shows that in case of a general fuzzy order, taking 1-cut means a loss of information.
 In the rest of this section we show that in case of lattice-type fuzzy order, the 1-cut represents the
 11 fuzzy order to a full extent.

We are going to introduce the notion of a lattice-type fuzzy order by requiring the existence of
 13 suprema and infima. Let thus \preceq be an \mathbf{L} -order on a set X that is equipped with an \mathbf{L} -equality \approx .
 For an \mathbf{L} -set A in X we define \mathbf{L} -sets $\mathcal{U}(A) \in L^X$ and $\mathcal{L}(A) \in L^X$ by

15
$$\mathcal{U}(A)(x) = \bigwedge_{y \in X} A(y) \rightarrow (y \preceq x)$$

and

17
$$\mathcal{L}(A)(x) = \bigwedge_{y \in X} A(y) \rightarrow (x \preceq y).$$

Basic rules of the semantics of fuzzy logic yield that $\mathcal{U}(A)(x)$ is the truth value of “for each $y \in X$:
 19 if y belongs to A then y is smaller than x ” and, similarly, $\mathcal{L}(A)(x)$ is the truth value of “for each
 $y \in X$: if y belongs to A then y is greater than x ”. Therefore, $\mathcal{U}(A)$ and $\mathcal{L}(A)$ may be called the
 21 upper cone of A and the lower cone of A , respectively. For more information about the operators \mathcal{U}
 and \mathcal{L} we refer to [5]. Before presenting the next definition recall that a singleton in $\langle X, \approx \rangle$ is an
 23 \mathbf{L} -set A in X which is compatible with \approx (i.e., $A(x) \otimes (x \approx y) \leq A(y)$) and satisfies, moreover, the
 following conditions:

- 25 (1) there exists $x \in X$ with $A(x) = 1$;
- (2) $A(x) \otimes A(y) \leq (x \approx y)$ for each $x, y \in X$.

27 If \mathbf{L} is the two-element Boolean algebra then singletons coincide with usual singletons, i.e. one-
 element sets.

1 **Definition 2** (Bělohlávek). For an \mathbf{L} -ordered set $\langle\langle X, \approx \rangle, \leqslant\rangle$ and $A \in L^X$ we define the \mathbf{L} -sets $\inf(A)$ and $\sup(A)$ in X by

$$3 \quad (\inf(A))(x) = (\mathcal{L}(A))(x) \wedge (\mathcal{U}\mathcal{L}(A))(x),$$

$$(\sup(A))(x) = (\mathcal{U}(A))(x) \wedge (\mathcal{L}\mathcal{U}(A))(x).$$

5 $\inf(A)$ and $\sup(A)$ are called the infimum and supremum of A , respectively. An \mathbf{L} -ordered set $\langle\langle X, \approx \rangle, \leqslant\rangle$ is said to be completely lattice \mathbf{L} -ordered if for any $A \in L^X$ both $\sup(A)$ and $\inf(A)$ are \approx -singletons.

7 **Remark 3.** The notions of infimum and supremum, as well as the notion of a completely lattice ordered set (complete lattice) are proper generalizations of the ordinary bivalent notions. Indeed, if \mathbf{L} is the two-element Boolean algebra, $(\inf(A))(x)$ is the truth value of the fact that x belongs to both the lower cone of A and the upper cone of the lower cone of A , i.e. x is the greatest lower bound of A ; similarly for $\sup(A)$.

13 In order to make sure that we do not deal with empty or artificial notions, we note that in the next section we will introduce the so-called fuzzy concept lattices (structures that are met in data analysis) which are natural examples of completely lattice fuzzy ordered sets. Moreover, we show that each completely lattice fuzzy ordered set is isomorphic to some fuzzy concept lattice. From this point of view, completely lattice fuzzy ordered sets form an important class of fuzzy ordered sets and their role is analogous to the role of complete lattices in ordinary relational systems. For example, each fuzzy ordered set can be embedded to a completely lattice fuzzy ordered set; the description of a minimal completion can be found in [6].

21 Having appropriate notion of a finite fuzzy set, one can obviously define the notion of a lattice type fuzzy order. We follow the common approach and consider a fuzzy set A in X finite if $\{x \mid A(x) > 0\}$ is a finite set. Then, a fuzzy ordered set $\langle\langle X, \approx \rangle, \leqslant\rangle$ is called a lattice fuzzy order if infima and suprema of any fuzzy set A in X are \approx -singletons. Clearly, each completely lattice fuzzy ordered set is also lattice fuzzy ordered.

The main result of our paper is contained in the following theorem.

27 **Theorem 3.** *Each lattice fuzzy ordered set is uniquely determined by its 1-cut. That is, if \leqslant_1 and \leqslant_2 are lattice \mathbf{L} -orders on $\langle X, \approx \rangle$ such that ${}^1\leqslant_1 = {}^1\leqslant_2$ then $\leqslant_1 = \leqslant_2$.*

29 **Proof.** Let \leqslant be a lattice fuzzy order on $\langle X, \approx \rangle$. Consider arbitrary $x, y \in X$. We need to show that there is only one way to extend ${}^1\leqslant$ to a complete lattice fuzzy order on $\langle X, \approx \rangle$. We distinguish two cases.

31 (1) Let x and y be comparable with respect to ${}^1\leqslant$, i.e. either $\langle x, y \rangle \in {}^1\leqslant$ or $\langle y, x \rangle \in {}^1\leqslant$. Without loss of generality let us assume $\langle x, y \rangle \in {}^1\leqslant$. By definition, $(x \leqslant y) = 1$. We show that the degree $(y \leqslant x)$ is uniquely given. Using antisymmetry, we get

$$35 \quad (y \leqslant x) = 1 \wedge (y \leqslant x) = (x \leqslant y) \wedge (y \leqslant x) \leqslant (x \approx y).$$

1 On the other hand, compatibility of \leq and \approx , and reflexivity of \leq and \approx yield

$$(x \approx y) = 1 \otimes (x \approx y) \otimes 1 = (x \leq x) \otimes (x \approx y) \otimes (x \approx x) \leq (y \leq x).$$

3 To sum up, $(y \leq x) = (x \approx y)$ showing the only one possibility for $(y \leq x)$. Therefore, if x and y are comparable w.r.t. $\overset{1}{\leq}$ then both $(x \leq y)$ and $(y \leq x)$ are uniquely determined.

5 (2) Let x and y be noncomparable with respect to $\overset{1}{\leq}$. Consider an \mathbf{L} -set A defined by $A(x) = 1$ and $A(y) = 1$. Since \leq is a lattice fuzzy order, there exists $\sup(A)$ which is a \approx -singleton. Therefore, there is some $z \in X$ with $\sup(A)(z) = 1$. Now, we have both $(x \leq z) = 1$ and $(y \leq z) = 1$. Indeed, $\sup(A)(z) = 1$ ensures that $\mathcal{U}(A)(z) = 1$. Since

$$\begin{aligned} \mathcal{U}(A)(z) &= \bigwedge_{u \in X} (A(u) \rightarrow (u \leq z)) \\ &\leq A(x) \rightarrow (x \leq z) = 1 \rightarrow (x \leq z) = (x \leq z), \end{aligned}$$

9 we get $1 \leq (x \leq z)$, thus $(x \leq z) = 1$. Due to symmetry of both of the cases we have $(y \leq z) = 1$ as well. Since x and z are comparable w.r.t. $\overset{1}{\leq}$, and also y and z are comparable w.r.t. $\overset{1}{\leq}$, case (1) implies that $(x \leq z)$, $(z \leq x)$, $(y \leq z)$, and $(z \leq y)$ are all determined uniquely. Therefore, it is sufficient to show that both $(x \leq y)$ and $(y \leq x)$ are uniquely determined by $(z \leq x)$ and $(z \leq y)$. Due to symmetry we show this fact only for $(x \leq y)$. Particularly, we shall show $(x \leq y) = (z \leq y)$ by proving both

$$(x \leq y) \leq (z \leq y)$$

15 and

$$(x \leq y) \geq (z \leq y).$$

17 “ \geq ”: Using $(x \leq z) = 1$ and transitivity of \leq we get

$$(z \leq y) = 1 \otimes (z \leq y) = (x \leq z) \otimes (z \leq y) \leq (x \leq y),$$

19 proving the first inequality.

21 “ \leq ”: Take the \mathbf{L} -set B in X such that $B(x) = 1$, $B(y) = 1$, and $B(z) = (z \leq y)$. We need the following Claim.

23 **Claim.** For \mathbf{L} -sets C, D in X with $D = C \cup \{a/z\}$ (i.e. D agrees with C with the possible exception $D(z) = a \vee C(z)$), we have that if $\mathcal{U}(C)(z) = 1$ and $\mathcal{L}\mathcal{U}(C)(z) = 1$ then $\mathcal{L}\mathcal{U}(D)(z) = 1$.

25 **Proof of Claim.** Clearly, it is sufficient to show $\mathcal{U}(D) = \mathcal{U}(C)$ (for then $\mathcal{L}\mathcal{U}(D)(z) = \mathcal{L}\mathcal{U}(C)(z) = 1$). Take an arbitrary $u \in X$. The inequality $\mathcal{U}(D)(u) \leq \mathcal{U}(C)(u)$ follows from the fact that $C \subseteq D$ since the operator \mathcal{U} is subsethood-reversing. On the other hand, we have

$$\begin{aligned} \mathcal{U}(D)(u) &= \left(\bigwedge_{w \neq z} D(w) \rightarrow (w \leq u) \right) \wedge ((C(z) \vee a) \rightarrow (z \leq u)) \\ &= \left(\bigwedge_{w \neq z} C(w) \rightarrow (w \leq u) \right) \wedge (C(z) \rightarrow (z \leq u)) \wedge (a \rightarrow (z \leq u)) \end{aligned}$$

$$\begin{aligned}
&= \left(\bigwedge_{w \in X} C(w) \rightarrow (w \leq u) \right) \wedge (a \rightarrow (z \leq u)) \\
&= \mathcal{U}(C)(u) \wedge (a \rightarrow (z \leq u)).
\end{aligned}$$

1 Therefore, to show $\mathcal{U}(C)(u) \leq \mathcal{U}(D)(u)$ it is enough to show

$$\mathcal{U}(C)(u) \leq (a \rightarrow (z \leq u)).$$

3 From $\mathcal{L}\mathcal{U}(C)(z) = 1$ we get

$$\mathcal{U}(C)(u) \leq (z \leq u).$$

5 Indeed,

$$1 = \mathcal{L}\mathcal{U}(C)(z) \leq \mathcal{U}(C)(u) \rightarrow (z \leq u).$$

7 Taking into account

$$(z \leq u) \leq (a \rightarrow (z \leq u)),$$

9 $\mathcal{U}(C)(u) \leq (a \rightarrow (z \leq u))$ follows.

Applying Claim (letting $C := A$, $D := B$, $a := (z \leq y)$), we get $\mathcal{L}\mathcal{U}(B)(z) = 1$. Now, since

$$\begin{aligned}
\mathcal{L}\mathcal{U}(B)(z) &= \bigwedge_{u \in X} \mathcal{U}(A)(u) \rightarrow (z \leq u) \\
&\leq \mathcal{U}(A)(y) \rightarrow (z \leq y),
\end{aligned}$$

11 we get

$$1 = \mathcal{U}(A)(y) \rightarrow (z \leq y),$$

13 whence

$$\mathcal{U}(A)(y) \leq (z \leq y).$$

15 Furthermore,

$$\begin{aligned}
\mathcal{U}(A)(y) &= (A(x) \rightarrow (x \leq y)) \wedge (A(y) \rightarrow (y \leq y)) \wedge (A(z) \rightarrow (z \leq y)) \\
&= (1 \rightarrow (x \leq y)) \wedge (1 \rightarrow 1) \wedge ((z \leq y) \rightarrow (z \leq y)) \\
&= (x \leq y).
\end{aligned}$$

Therefore,

$$17 \quad (x \leq y) = \mathcal{U}(A)(y) \leq (z \leq y),$$

proving the required inequality. The proof is finished. \square

With appropriate modification of a result from [6] we get the following theorem.

1 **Theorem 4.** For an \mathbf{L} -ordered set $\mathbf{X} = \langle \langle X, \approx \rangle, \leq \rangle$, the relation $^1\leq$ is an order on X . Moreover, if \mathbf{X} is (completely) lattice \mathbf{L} -ordered then $^1\leq$ is a (complete) lattice order on X .

3 **Proof.** Denote $^1\leq$ by \subseteq . Reflexivity of \subseteq follows from reflexivity of \leq . Antisymmetry of \subseteq : $x \subseteq y$ and $y \subseteq x$ implies $(x \leq y) = 1$ and $(y \leq x) = 1$. Antisymmetry of \leq thus yields $(x \approx y) = 1$. Since \approx is
5 an \mathbf{L} -equality, we conclude $x = y$. If $x \subseteq y$ and $y \subseteq z$, then $(x \leq y) = 1$ and $(y \leq z) = 1$, therefore

$$1 = (x \leq y) \otimes (y \leq z) \leq (x \leq z),$$

7 whence $(x \leq z) = 1$, i.e. $x \subseteq z$, by transitivity of \leq .

Let \mathbf{X} be completely lattice \mathbf{L} -ordered and let a (finite) subset $A \subseteq X$ be given. Denote by A'
9 the \mathbf{L} -set in X corresponding to A , i.e. $A'(x) = 1$ for $x \in A$ and $A'(x) = 0$ for $x \notin A$. We show that there exists a supremum $\bigvee A$ of A in $\langle X, \subseteq \rangle$ (the case of infimum is dual). Since \mathbf{X} is completely
11 lattice \mathbf{L} -ordered, $\sup(A')$ is a \approx -singleton in $\langle X, \approx \rangle$. Denote by x^* the element of X such that $(\sup(A'))(x^*) = 1$. Since

$$13 \quad (\sup(A'))(x^*) = (\mathcal{U}(A'))(x^*) \wedge (\mathcal{L}\mathcal{U}(A'))(x^*),$$

we have both $(\mathcal{U}(A'))(x^*) = 1$ and $(\mathcal{L}\mathcal{U}(A'))(x^*) = 1$. From the former we have

$$15 \quad \bigwedge_{x \in X} (A'(x) \rightarrow (x \leq x^*)) = 1,$$

i.e. $A'(x) \leq (x \leq x^*)$ by adjointness. Therefore, $(x \leq x^*) = 1$ for any $x \in X$ such that $A'(x) = 1$ (i.e.
17 $x \in A$). Therefore, x^* belongs to the upper cone (w.r.t. \subseteq) of A . In a similar way we can show that $(\mathcal{L}\mathcal{U}(A'))(x^*) = 1$ implies that x^* belongs to the lower cone of the upper cone (cones w.r.t. \subseteq)
19 of A . Thus, x^* is the supremum of A w.r.t. \subseteq . \square

21 **Remark 4.** The 1-cut of the fuzzy order from Example 1 is a lattice order. This shows that the condition on \leq from Theorem 3 cannot be weakened by requiring only that the 1-cut of \leq be a lattice.

23 3. Representative example and discussion: fuzzy concept lattices

The role of lattices and complete lattices in mathematics and its applications is well known. In
25 this section, we show that completely lattice fuzzy ordered sets appear in a natural way in the context of analysis of fuzzy data. Particularly, the set of all formal fuzzy concepts which is the set
27 of all natural concepts (clusters) hidden in the input data as defined in formal concept analysis is a completely lattice fuzzy ordered set. Moreover, this example is representative for completely lattice
29 fuzzy ordered sets.

Let X and Y denote sets interpreted as a set of objects and a set of attributes, respectively. Let
31 I be an \mathbf{L} -relation between X and Y , the degree $I(x, y)$ being interpreted as the truth degree to which the object $x \in X$ has the attribute $y \in Y$. We define operators (induced by I) $^\uparrow : L^X \rightarrow L^Y$ and
33 $^\downarrow : L^Y \rightarrow L^X$ assigning to any \mathbf{L} -set $A \in L^X$ the \mathbf{L} -set $A^\uparrow \in L^Y$ defined by

$$A^\uparrow(y) = \bigwedge_{x \in X} A(x) \rightarrow I(x, y)$$

and to any \mathbf{L} -set $B \in L^Y$ the \mathbf{L} -set $B^\downarrow \in L^X$ defined by

$$B^\downarrow(x) = \bigwedge_{y \in Y} B(y) \rightarrow I(x, y).$$

Described verbally, A^\uparrow is the fuzzy set of all attributes common to all objects from A and B^\downarrow is the fuzzy set of all objects sharing all attributes from B . A pair $\langle A, B \rangle$ of a fuzzy set $A \in L^X$ of objects and a fuzzy set $B \in L^Y$ of attributes is called a formal concept in $\langle X, Y, I \rangle$ if $A^\uparrow = B$ and $B^\downarrow = A$. The verbal description of the condition $A^\uparrow = B$ and $B^\downarrow = A$, i.e. saying that B is the collection of all attributes common to all objects from A and that A is the collection of all objects sharing all attributes from B , is exactly the way the notion of a concept was defined in the so-called traditional logic (this conception was elaborated in Port-Royal, see [1]). The set $\mathcal{B}(X, Y, I) = \{ \langle A, B \rangle \mid A^\uparrow = B, B^\downarrow = A \}$ of all formal fuzzy concepts in $\langle X, Y, I \rangle$ may be therefore regarded as the set of all natural concepts hidden in the input data described by $\langle X, Y, I \rangle$. One may introduce the following \mathbf{L} -relations on $\mathcal{B}(X, Y, I)$:

$$(\langle A_1, B_1 \rangle \approx \langle A_2, B_2 \rangle) = \bigwedge_{x \in X} A_1(x) \leftrightarrow A_2(x)$$

and

$$(\langle A_1, B_1 \rangle \leq \langle A_2, B_2 \rangle) = \bigwedge_{x \in X} A_1(x) \rightarrow A_2(x).$$

It can be shown that

$$(\langle A_1, B_1 \rangle \approx \langle A_2, B_2 \rangle) = \bigwedge_{y \in Y} B_1(y) \leftrightarrow B_2(y)$$

and

$$(\langle A_1, B_1 \rangle \leq \langle A_2, B_2 \rangle) = \bigwedge_{y \in Y} B_2(y) \rightarrow B_1(y).$$

It is clear that $(\langle A_1, B_1 \rangle \approx \langle A_2, B_2 \rangle)$ may be interpreted as the degree of equality of fuzzy concepts $\langle A_1, B_1 \rangle$ and $\langle A_2, B_2 \rangle$. Likewise, $(\langle A_1, B_1 \rangle \leq \langle A_2, B_2 \rangle)$ may be interpreted as the degree to which $\langle A_1, B_1 \rangle$ is a subconcept of $\langle A_2, B_2 \rangle$ in that each object covered by $\langle A_1, B_1 \rangle$ is also covered by $\langle A_2, B_2 \rangle$. Fuzzy concept lattices were investigated in a series of papers (see [5] for a survey). Particularly, the following characterization was shown in [6]:

Theorem 5. Let $\langle X, Y, I \rangle$ be an \mathbf{L} -context. (1) $(\langle \mathcal{B}(X, Y, I), \approx, \leq \rangle)$ is completely lattice \mathbf{L} -ordered set in which infima and suprema can be described as follows: for an \mathbf{L} -set \mathcal{M} in $\mathcal{B}(X, Y, I)$ we have

$${}^1 \text{inf}(\mathcal{M}) = \left\{ \left\langle \bigcap_X \mathcal{M}, \left(\bigcap_X \mathcal{M} \right)^\uparrow \right\rangle \right\} = \left\{ \left\langle \left(\bigcup_Y \mathcal{M} \right)^\downarrow, \left(\bigcup_Y \mathcal{M} \right)^{\downarrow\uparrow} \right\rangle \right\}, \quad (1)$$

$${}^1\text{sup}(\mathcal{M}) = \left\{ \left\langle \left(\bigcap_Y \mathcal{M} \right)^\downarrow, \bigcap_Y \mathcal{M} \right\rangle \right\} = \left\{ \left\langle \left(\bigcup_X \mathcal{M} \right)^{\uparrow\downarrow}, \left(\bigcup_X \mathcal{M} \right)^{\uparrow} \right\rangle \right\}. \quad (2)$$

(2) Moreover, a completely lattice \mathbf{L} -ordered set $\mathbf{V} = \langle \langle V, \approx \rangle, \leq \rangle$ is isomorphic to $\langle \langle \mathcal{B}(X, Y, I), \approx \rangle, \leq \rangle$ iff there are mappings $\gamma: X \times L \rightarrow V$, $\mu: Y \times L \rightarrow V$, such that $\gamma(X \times L)$ is $\{0, 1\}$ -supremally dense in \mathbf{V} , $\mu(Y \times L)$ is $\{0, 1\}$ -infimally dense in \mathbf{V} , and $((a \otimes b) \rightarrow I(x, y)) = (\gamma(x, a) \leq \mu(y, b))$ for all $x \in X$, $y \in Y$, $a, b \in L$. In particular, \mathbf{V} is isomorphic to $\mathcal{B}(V, V \leq)$.

Note that for \mathbf{X} being a completely lattice \mathbf{L} -ordered set, $L' \subseteq L$, we say that a subset $K \subseteq X$ is L' -infimally dense in \mathbf{X} (L' -supremally dense in \mathbf{X}) if for each $x \in X$ there is some $A \in L'^X$ such that $A(x) = 0$ for all $x \notin K$ and $(\inf(A))(x) = 1$ ($(\sup(A))(x) = 1$). For the bivalent case and $\{0, 1\}$ -infimal density (usually termed simply infimal density) of K means that each element of X is an infimum of some subset of K (similarly for supremal density). Furthermore, two \mathbf{L} -ordered sets $\langle \langle V_1, \approx_1 \rangle, \leq_1 \rangle$ and $\langle \langle V_2, \approx_2 \rangle, \leq_2 \rangle$ are considered isomorphic if there is a bijective mapping $h: V_1 \rightarrow V_2$ such that $(u \approx_1 v) = (h(u) \approx_2 h(v))$ and $(u \leq_1 v) = (h(u) \leq_2 h(v))$ for each $u, v \in V_1$. Note also that for a fuzzy set A in a completely lattice fuzzy ordered set, both the infimum $\inf(A)$ and the supremum $\sup(A)$ are singletons and are thus uniquely determined by their 1-cuts ${}^1\inf(A)$ and ${}^1\sup(A)$. This fact justifies the description of infima and suprema in Theorem 5.

Due to Theorem 4, $\mathcal{B}(X, Y, I)$ equipped with ${}^1\leq$ is a complete lattice. It turned out that \approx is quite a useful \mathbf{L} -relation on $\mathcal{B}(X, Y, I)$. In fact, it models the intuitively very natural notion of a similarity of formal fuzzy concepts. Moreover, it was shown in [2] that each a -cut ${}^a\approx$ of \approx is a tolerance relation (i.e. reflexive and symmetric) which is compatible with the complete lattice structure of $\langle \mathcal{B}(X, Y, I), {}^1\leq \rangle$. This makes it possible to factorize the original fuzzy concept lattice $\mathcal{B}(X, Y, I)$ and to consider a simplified structure $\mathcal{B}(X, Y, I)/{}^a\approx$ (the factor lattice modulo ${}^a\approx$). This is important from the point of view of applications since the fuzzy concept lattice of a given fuzzy context is usually quite large and so techniques for its simplification are needed.

While the crisp hierarchy ${}^1\leq$ on $\mathcal{B}(X, Y, I)$ is of crucial importance for applications (it models the crisp subconcept-superconcept relationship which is used for visualization of the fuzzy concept lattice), we were not sure if there is any substantial gain in using \leq on $\mathcal{B}(X, Y, I)$. Experience shows that: First, \leq contains too much an information which is hardly graspable by human mind. Second, the 1-cut ${}^1\leq$ of \leq is much more easier to interpret than \leq . This is probably due to the fact that contrary to similarity which people expect to be fuzzy in essence (subconcept–superconcept) hierarchy is more expected to be crisp. However, without no additional insight, disregarding \leq (and considering only ${}^1\leq$) may mean a loss of information. Now, Theorem 3 presents a mathematical solution to this dilemma: $\langle \mathcal{B}(X, Y, I), \approx, \leq \rangle$ is uniquely determined by $\langle \mathcal{B}(X, Y, I), \approx, {}^1\leq \rangle$, i.e. by what is intuitively considered a useful information derived from the input data $\langle X, Y, I \rangle$.

Remark 5. As mentioned above, fuzzy concept lattices are representative examples of completely lattice fuzzy ordered sets. Another representative examples are sets of fixed points of fuzzy closure operators [4]: A mapping $C: L^X \rightarrow L^X$ is called an \mathbf{L} -closure operator if

- (1) $S(A_1, A_2) \leq S(C(A_1), C(A_2))$,
- (2) $A \subseteq C(A)$,
- (3) $C(A) = C(C(A))$.

1 For an **L**-closure operator C , the set

$$\mathcal{S}_C = \{A \in L^X \mid A = C(A)\},$$

3 equipped with \approx and S , defined by

$$(A \approx B) = \bigwedge_{x \in X} (A(x) \leftrightarrow B(x))$$

5 and

$$S(A, B) = \bigwedge_{x \in X} (A(x) \rightarrow B(x)),$$

7 the structure $\langle\langle \mathcal{S}_C, \approx \rangle, S\rangle$ is a completely lattice **L**-ordered set and conversely, each completely lattice
L-ordered set is isomorphic to some \mathcal{S}_C (see also [5]). A trivial example of an **L**-closure operator
 9 is the identity mapping ($C(A) = A$) showing that the set L^X of all **L**-sets in a given set X equipped
 with \approx and S is a completely lattice fuzzy ordered set.

11 **Remark 6.** With respect to the above discussion, Theorem 3 may be considered a positive result: If
 one ends up with a lattice-type fuzzy ordered set (like if one obtains a fuzzy concept lattice from
 13 object-attribute data), one may restrict attention to its 1-cut. Due to Theorem 3, it contains all the
 information about the hierarchy modeled by the fuzzy ordered set. Except for its above-discussed
 15 practical impact, Theorem 3 shows an interesting mathematical result. On the other hand, since
 all other degrees are determined by the 1-cut, it may be argued that the modeling capability of
 17 lattice-type fuzzy order is restricted. Going through the proofs, it is clear that a lot of information
 contained in \preceq is already present in \approx . This somewhat weakens the restricted modeling capability
 19 viewpoint since changing \approx can be considered as changing a parameter which leads to a whole class
 of lattice-type fuzzy orders (with a common 1-cut).

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