

A Note on the Extension Principle

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It is shown that a function obtained by Zadeh's extension principle preserves similarity. © 200? Academic Press

1. INTRODUCTION

A fuzzy set in a universe set X is a mapping $A : X \rightarrow L$ where L is the support of an appropriate structure \mathbf{L} of truth values (usually $L = [0, 1]$). In [5], Zadeh proposed a so called extension principle which become an important tool in fuzzy set theory and applications. The idea is that each function $f : X \rightarrow Y$ induces a corresponding function $\bar{f} : L^X \rightarrow L^Y$ (i.e. \bar{f} is a function mapping fuzzy sets in X to fuzzy sets in Y) defined for each fuzzy set A in X by

$$\bar{f}(A)(y) = \bigvee_{x \in X, f(x)=y} A(x).$$

The function \bar{f} is said to be obtained from f by the extension principle. Therefore, given a precise functional dependence expressed by f , one can compute f (in fact, \bar{f}) even when the argument of f is described only approximately (and expressed by a fuzzy set). The aim of this note is to show that the function \bar{f} preserves (in a natural way) similarity of fuzzy sets.

A general structure of truth values is represented by complete residuated lattices introduced into the context of fuzzy sets and fuzzy logic by Goguen [2] (see also [3, 4] for further justification from the point of view of logic). Recall that a complete residuated lattice is an algebra $\mathbf{L} = \langle L, \otimes, \rightarrow, \wedge, \vee, 0, 1 \rangle$ where (i) $\langle L, \wedge, \vee, 0, 1 \rangle$ is a complete lattice with

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the least element 0 and the greatest element 1, (ii) $\langle L, \otimes, 1 \rangle$ is a commutative monoid, (iii) \otimes and \rightarrow satisfy the adjointness property, i.e. $x \leq y \rightarrow z$ iff $x \otimes y \leq z$ holds. The most applied complete residuated lattices are those with $L = [0, 1]$ and the following structures: Łukasiewicz ($a \otimes b = \max(a + b - 1, 0)$, $a \rightarrow b = \min(1 - a + b, 1)$), Gödel ($a \otimes b = \min(a, b)$, $a \rightarrow b = 1$ if $a \leq b$ and $= b$ otherwise), and product ($a \otimes b = a \cdot b$, $a \rightarrow b = 1$ if $a \leq b$ and $= b/a$ otherwise). For further properties of residuated lattices we refer to [2, 3, 4]. Note that if $L = [0, 1]$ then \otimes is a t -norm [3]. Moreover, each left-continuous t -norm \otimes makes L into a complete residuated lattice putting $a \rightarrow b = \bigvee \{c \mid a \otimes c \leq b\}$.

2. RESULTS

We are going to show that the function \bar{f} preserves similarity. Recall that a similarity relation in a set U is a binary fuzzy relation $E : U \times U \rightarrow L$ which is reflexive (i.e. $E(x, x) = 1$), symmetric (i.e. $E(x, y) = E(y, x)$), and transitive (i.e. $E(x, y) \otimes E(y, z) \leq E(x, z)$). There is a natural way to measure similarity between fuzzy sets. It is expressed in the following proposition (see e.g. [1]).

PROPOSITION 2.1. *Let X be a set. The binary fuzzy relation E_X defined on L^X by $E_X(A_1, A_2) = \bigwedge_{x \in X} A_1(x) \leftrightarrow A_2(x)$ is a similarity relation in L^X .*

Here \leftrightarrow denotes the biresiduum operation defined by $x \leftrightarrow y = (x \rightarrow y) \wedge (y \rightarrow x)$. Note that if interpreted on the linguistic level, two fuzzy sets are similar iff they contain the same elements.

In the case of the above noted Łukasiewicz, Gödel, and product t -norms we obtain

$$\begin{aligned} E(A_1, A_2) &= \inf_{x \in X} 1 - |A_1(x) - A_2(x)| \quad (\text{Łukasiewicz}) \\ E(A_1, A_2) &= \inf_{x \in X} \min(A_1(x), A_2(x)) \quad (\text{Gödel}) \\ E(A_1, A_2) &= \inf_{x \in X} \min(A_1(x)/A_2(x), A_2(x)/A_1(x)) \quad (\text{product}) \end{aligned}$$

where we put $0/0 = 1$ and $a/0 = \infty$ for $a \neq 0$.

THEOREM 2.1. *Let \bar{f} be the function obtained from $f : X \rightarrow Y$ by the extension principle. Then for every fuzzy sets A_1, A_2 in X it holds that $E_X(A_1, A_2) \leq E_Y(\bar{f}(A_1), \bar{f}(A_2))$.*

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Proof. By the definition of \bar{f} we have to show that

$$E_X(A_1, A_2) \leq \bigwedge_{y \in Y} \left(\bigvee_{x \in X, f(x)=y} A_1(x) \leftrightarrow \bigvee_{x \in X, f(x)=y} A_2(x) \right)$$

which holds iff for each $y \in Y$ it holds

$$E_X(A_1, A_2) \leq \bigvee_{x \in X, f(x)=y} A_1(x) \leftrightarrow \bigvee_{x \in X, f(x)=y} A_2(x).$$

To show this, we have to check that the left side of the inequality is less or equal to both $\bigvee_{x \in X, f(x)=y} A_1(x) \rightarrow \bigvee_{x \in X, f(x)=y} A_2(x)$ and $\bigvee_{x \in X, f(x)=y} A_2(x) \rightarrow \bigvee_{x \in X, f(x)=y} A_1(x)$. Due to symmetry we proceed for the first case only:

$$E_X(A_1, A_2) \leq \bigvee_{x \in X, f(x)=y} A_1(x) \rightarrow \bigvee_{x \in X, f(x)=y} A_2(x)$$

is (by the adjointness property of residuated lattices) equivalent to

$$\left(\bigvee_{x \in X, f(x)=y} A_1(x) \right) \otimes E_X(A_1, A_2) \leq \bigvee_{x \in X, f(x)=y} A_2(x)$$

which is true since

$$\begin{aligned} & \left(\bigvee_{x \in X, f(x)=y} A_1(x) \right) \otimes E_X(A_1, A_2) = \\ & = \bigvee_{x \in X, f(x)=y} (A_1(x) \otimes E_X(A_1, A_2)) \leq \\ & \leq \bigvee_{x \in X, f(x)=y} (A_1(x) \otimes (A_1(x) \rightarrow A_2(x))) \leq \bigvee_{x \in X, f(x)=y} A_2(x). \end{aligned}$$

The assertion is proved. \blacksquare

Note that the result says that the truth value of the proposition "if A_1 and A_2 are similar then $\bar{f}(A_1)$ and $\bar{f}(A_2)$ are similar" is 1 (full truth).

The following corollary is immediate.

COROLLARY 2.1. *Under the notation of Theorem 2.1, if f is an injective function (i.e. $x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$) then $E_X(A_1, A_2) = E_Y(\bar{f}(A_1), \bar{f}(A_2))$.*

The extension principle is sometimes formulated to extend a function $f : X_1 \times \cdots \times X_n \rightarrow Y$ into $\bar{f} : L^{X_1} \times \cdots \times L^{X_n} \rightarrow L^Y$ by

$$\bar{f}(A_1, \dots, A_n)(y) = \bigvee_{x_1 \in X_1, \dots, x_n \in X_n, f(x_1, \dots, x_n) = y} A_1(x_1) \otimes \cdots \otimes A_n(x_n)(1)$$

for A_1, \dots, A_n being fuzzy sets in X_1, \dots, X_n , respectively. For such a case we have the following assertion.

THEOREM 2.2. *Let \bar{f} be the function obtained from $f : X_1 \times \cdots \times X_n \rightarrow Y$ by (1). Then for every fuzzy sets $A_1, B_1 \in L^{X_1}, \dots, A_n, B_n \in L^{X_n}$ it holds that*

$$E_{X_1}(A_1, B_1) \otimes \cdots \otimes E_{X_n}(A_n, B_n) \leq E_Y(\bar{f}(A_1, \dots, A_n), \bar{f}(B_1, \dots, B_n)).$$

Proof. The assertion is a consequence of Theorem 2.1. Namely, putting $X = X_1 \times \cdots \times X_n$ and $A = A_1 \otimes \cdots \otimes A_n$, $B = B_1 \otimes \cdots \otimes B_n$ where $A \in L^X$ is defined by $A(x_1, \dots, x_n) = A_1(x_1) \otimes \cdots \otimes A_n(x_n)$ (and similarly for B), Theorem 2.1 yields

$$E_X(A, B) \leq E_Y(\bar{f}(A_1, \dots, A_n), \bar{f}(B_1, \dots, B_n)).$$

To prove the assertion, it is therefore sufficient to prove

$$E_{X_1}(A_1, B_1) \otimes \cdots \otimes E_{X_n}(A_n, B_n) \leq E_X(A, B)$$

which holds iff for every $\mathbf{x} = \langle x_1, \dots, x_n \rangle \in X_1 \times \cdots \times X_n$, $\mathbf{y} = \langle y_1, \dots, y_n \rangle \in Y_1 \times \cdots \times Y_n$, it holds

$$E_{X_1}(A_1, B_1) \otimes \cdots \otimes E_{X_n}(A_n, B_n) \leq A(\mathbf{x}) \rightarrow B(\mathbf{x}) \wedge B(\mathbf{y}) \rightarrow A(\mathbf{y}).$$

To check this inequality we verify

$$E_{X_1}(A_1, B_1) \otimes \cdots \otimes E_{X_n}(A_n, B_n) \leq A(\mathbf{x}) \rightarrow B(\mathbf{x})$$

(due to symmetry we omit checking the second inequality which is to verify) which is equivalent to

$$A(\mathbf{x}) \otimes E_{X_1}(A_1, B_1) \otimes \cdots \otimes E_{X_n}(A_n, B_n) \leq B(\mathbf{x}).$$

The last inequality is, indeed, true since

$$\begin{aligned} A(\mathbf{x}) \otimes E_{X_1}(A_1, B_1) \otimes \cdots \otimes E_{X_n}(A_n, B_n) &= \\ &= A_1(x_1) \otimes E_{X_1}(A_1, B_1) \otimes \cdots \otimes A_n(x_n) \otimes E_{X_n}(A_n, B_n) \leq \\ &\leq A_1(x_1) \otimes (A_1(x_1) \rightarrow B_1(x_1)) \otimes \cdots \otimes A_n(x_n) \otimes (A_n(x_n) \rightarrow B_n(x_n)) \leq \\ &\leq B_1(x_1) \otimes \cdots \otimes B_n(x_n) = B(\mathbf{x}). \end{aligned}$$

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Extension principle is used mainly in situations where no precise description of the input data is available, e.g. if linguistic variable is used to describe the inputs. In such a case, using extension principle we can make our model “tolerant to imprecision” of that kind. Our results say that if we do that, the exact description of the imprecise inputs (exact shape of membership function) does not matter.

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