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A note on variable threshold concept lattices: Threshold-based operators are reducible to classical concept-forming operators

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Abstract

The present paper deals with formal concept analysis of data with fuzzy attributes. We clarify several points of a new approach of [S.Q. Fan, W.X. Zhang, Variable threshold concept lattice, Inf. Sci., accepted for publication] which is based on using thresholds in concept-forming operators. We show that the extent- and intent-forming operators from [S.Q. Fan, W.X. Zhang, Inf. Sci., accepted for publication] can be defined in terms of basic fuzzy set operations and the original operators as introduced and studied e.g. in [R. Belohlavek, Fuzzy Galois connections, Math. Logic Quarterly 45 (4) (1999) 497–504; R. Belohlavek, Concept lattices and order in fuzzy logic, Ann. Pure Appl. Logic 128 (2004) 277–298; S. Pollandt, Fuzzy Begriffe, Springer-Verlag, Berlin/Heidelberg, 1997]. As a consequence, main properties of the new operators from [S.Q. Fan, W.X. Zhang, Inf. Sci., accepted for publication], including the properties studied in [S.Q. Fan, W.X. Zhang, Inf. Sci., accepted for publication], can be obtained as consequences of the original operators from [R. Belohlavek, 1999; R. Belohlavek, 2004; S. Pollandt, 1997].

Keywords: Concept lattice; Galois connection; Fuzzy logic; Formal concept analysis

1. Introduction

Inspired by previous work on formal concept analysis of data with fuzzy attributes, the authors in [10] introduce new pairs of extent- and intent-forming operators and show basic relationships between the corresponding concept lattices, i.e. the set of fixpoints of these operators. The main contribution of the authors is that they showed a way to incorporate the idea of thresholds into the definition of extent- and intent-forming operators in such a way that some desirable properties remain available. In [10], the idea of thresholds was taken from [9] but a particular case of this idea (namely, threshold being equal to 1) appeared in previous

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papers already, e.g. [5,6,16,8]. Thresholds play the role of parameters which enable us to reduce the number of formal concepts from a given data table with fuzzy attributes. The results of [10] are therefore important from the point of view of extraction of patterns from data.

The aim of this note is to show that the extent- and intent-forming operators from [10] can be defined in terms of basic fuzzy set operations and the original operators as introduced and studied e.g. in [1,3,17]. As a consequence, several properties of the new operators from [10], including the properties studied in [10], can be obtained as consequences of the original operators from [1,3,17].

2. Results

2.1. Preliminaries

We first briefly recall the necessary notions (we refer to [10] and to [1,3,2,15] for further details). As a structure of truth degrees, we use an arbitrary complete residuated lattice $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \to, 0, 1\rangle$, i.e. $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice with 0 and 1 being the least and greatest element of L, respectively (for instance, L is [0,1], a finite chain, etc.); $\langle L, \otimes, 1\rangle$ is a commutative monoid (i.e. $\otimes$ is commutative, associative, and $a \otimes 1 = 1 \otimes a = a$ for each $a \in L$); and $\otimes$ and $\to$ satisfy so-called adjointness property, i.e. $a \otimes b \leqslant c$ iff $a \leqslant b \to c$ for each $a, b, c \in L$. Well-known examples of complete residuated lattices used often in fuzzy logic applications include complete residuated lattices on the unit interval $L = [0, 1]$. Three important pairs of adjoint operators $\otimes$ and $\to$ are

$$\text{Łukasiewicz:}\quad \begin{aligned} a \otimes b &= \max(a + b - 1, 0),\\ a \to b &= \min(1 - a + b, 1),\end{aligned}$$

$$\text{Gödel:}\quad \begin{aligned} a \otimes b &= \min(a, b),\\ a \to b &= \begin{cases} 1 & \text{if } a \leqslant b,\\ b & \text{otherwise},\end{cases}\end{aligned}$$

$$\text{Goguen(product):}\quad \begin{aligned} a \otimes b &= a \cdot b,\\ a \to b &= \begin{cases} 1 & \text{if } a \leqslant b,\\ \frac{b}{a} & \text{otherwise.}\end{cases}\end{aligned}$$

Another example is provided by finite chains, e.g. $L = \{a_0 = 0, a_1, \ldots, a_n = 1\} \subseteq [0, 1]$ $(a_0 < \cdots < a_n)$. A Łukasiewicz chain is given by $a_k \otimes a_l = a_{\max(k+l-n,0)}$ and $a_k \to a_l = a_{\min(n-k+l,n)}$. A Gödel chain is given by $a_k \otimes a_l = a_{\min(k,l)}$ and $a_k \to a_l = 1$ if $k \leqslant l$ and $= a_l$ otherwise.

Note that in [10], the authors do not require commutativity of $\otimes$, i.e. commutativity of fuzzy conjunction. However, the authors do not provide any motivation for having non-commutative fuzzy conjunction. Non-commutativity of $\otimes$ appears as an implicit possibility in [10]. We do proceed with a commutative fuzzy conjunction. Note that a thorough study of non-commutative concept lattices and related structures was performed by Georgescu and Popescu, see e.g. [12–14].

By L^U we denote the set of all fuzzy sets in universe U, i.e. $L^U = \{A \mid A \text{ is a mapping of } U \text{ to } L\}$; by 2^U we denote the set of all ordinary subsets of U, and by abuse of notation we sometimes identify ordinary subsets of U with crisp fuzzy sets from L^U, i.e. with those $A \in L^U$ for which $A(u) = 0$ or $A(u) = 1$ for each $u \in U$. A formal fuzzy context can be identified with a triplet $\langle X, Y, I\rangle$ where X is a non-empty set of objects, Y is a non-empty set of attributes, and I is a fuzzy relation between X and Y, i.e. $I : X \times Y \to L$. For $y \in X$ and $y \in Y$, a degree $I(x, y) \in L$ is interpreted as a degree to which object x has attribute y. A formal context $\langle X, Y, I\rangle$ can be seen as a data table with fuzzy attributes with rows and columns corresponding to objects and attributes, and table entries filled with truth degrees $I(x, y)$.

2.2. Galois connections and concept lattices defined by thresholds

For fuzzy sets $A \in L^X$ and $B \in L^Y$, consider fuzzy sets $A^{\uparrow} \in L^Y$ and $B^{\downarrow} \in L^X$ (denoted also $A^{\uparrow_I}$ and $B^{\downarrow_I}$ to make I explicit) defined by

$$A^{\uparrow}(y) = \wedge_{x\in X}(A(x) \to I(x,y)), \tag{1}$$
$$B^{\downarrow}(x) = \wedge_{y\in Y}(B(y) \to I(x,y)). \tag{2}$$

Operators $^{\uparrow}$ and $^{\downarrow}$ were introduced in [1,3,17] as a direct generalization of the corresponding operators induced by ordinary formal contexts, i.e. data tables with bivalent attributes, see [11]. Using basic rules of predicate fuzzy logic, $A^{\uparrow}$ is a fuzzy set of all attributes common to all objects from A, and $B^{\downarrow}$ is a fuzzy set of all objects sharing all attributes for which it is very true that they belong to B. The set

$$\mathscr{B}(X,Y,I) = \{\langle A,B\rangle | A^{\uparrow} = B,\ B^{\downarrow} = A\}$$

of all fixpoints of $\langle^{\uparrow},^{\downarrow}\rangle$ is called a fuzzy concept lattice of $\langle X,Y,I\rangle$; elements $\langle A,B\rangle \in \mathscr{B}(X,Y,I)$ will be called formal concepts of $\langle X,Y,I\rangle$; A and B are called the extent and intent of $\langle A,B\rangle$, respectively. Under a partial order $\leqslant$ defined on $\mathscr{B}(X,Y,I)$ by

$$\langle A_1,B_1\rangle \leqslant \langle A_2,B_2\rangle \quad \text{iff } A_1 \subseteq A_2,$$

$\mathscr{B}(X,Y,I)$ happens to be a complete lattice and we refer to [2] for results describing the structure of $\mathscr{B}(X,Y,I)$. Note that $\mathscr{B}(X,Y,I)$ is the basic structure used for formal concept analysis of the data table represented by $\langle X,Y,I\rangle$.

In addition to the pair of operators $^{\uparrow} : L^X \to L^Y$ and $^{\downarrow} : L^Y \to L^X$, the authors in [10] define pairs of operators (we keep the notation of [10]) $^{*} : 2^X \to 2^Y$ and $^{*} : 2^Y \to 2^X$, $^{\Box} : 2^X \to L^Y$ and $^{\Box} : L^Y \to 2^X$, and $^{\Diamond} : L^X \to 2^Y$ and $^{\Diamond} : 2^Y \to L^X$, as follows. Let δ be an arbitrary truth degree from L (δ plays a role of a threshold). For $A \in L^X$, $C \in 2^X$, $B \in L^Y$, $D \in 2^Y$ define $C^* \in 2^Y$ and $D^* \in 2^X$ by

$$C^* = \{y \in Y | \wedge_{x\in X}(C(x) \to I(x,y)) \geqslant \delta\}, \tag{3}$$
$$D^* = \{x \in X | \wedge_{y\in Y}(D(y) \to I(x,y)) \geqslant \delta\}, \tag{4}$$

$C^{\Box} \in L^Y$ and $B^{\Box} \in 2^X$ by

$$C^{\Box}(y) = \delta \to \wedge_{x\in C} I(x,y), \tag{5}$$
$$B^{\Box} = \{x \in X | \wedge_{y\in Y}(B(y) \to I(x,y)) \geqslant \delta\}, \tag{6}$$

and $A^{\Diamond} \in 2^Y$ and $D^{\Diamond} \in L^X$ by

$$A^{\Diamond} = \{y \in Y | \wedge_{x\in X}(A(x) \to I(x,y)) \geqslant \delta\}, \tag{7}$$
$$D^{\Diamond}(x) = \delta \to \wedge_{y\in D} I(x,y), \tag{8}$$

for each $x \in X$, $y \in Y$.

Denote now the corresponding set of fixpoints of these pairs of operators by

$$\mathscr{B}(X_*,Y_*,I) = \{\langle A,B\rangle \in 2^X \times 2^Y | A^* = B, B^* = A\},$$
$$\mathscr{B}(X_{\Box},Y_{\Box},I) = \{\langle A,B\rangle \in 2^X \times L^Y | A^{\Box} = B, B^{\Box} = A\},$$
$$\mathscr{B}(X_{\Diamond},Y_{\Diamond},I) = \{\langle A,B\rangle \in L^X \times 2^Y | A^{\Diamond} = B, B^{\Diamond} = A\},$$
$$\mathscr{B}(X_{\uparrow},Y_{\downarrow},I) = \{\langle A,B\rangle \in L^X \times L^Y | A^{\uparrow} = B, B^{\downarrow} = A\} \quad (= \mathscr{B}(X,Y,I)).$$

As usual, $\mathrm{Ext}(X_*,Y_*,I)$ and $\mathrm{Int}(X_*,Y_*,I)$ denote the sets of extents and intents of $\mathscr{B}(X_*,Y_*,I)$, i.e.

$$\mathrm{Ext}(X_*,Y_*,I) = \{A \in L^X | \langle A,B\rangle \in \mathscr{B}(X_*,Y_*,I) \text{ for some } B \in L^Y\},$$
$$\mathrm{Int}(X_*,Y_*,I) = \{B \in L^Y | \langle A,B\rangle \in \mathscr{B}(X_*,Y_*,I) \text{ for some } A \in L^X\}.$$

For $\mathscr{B}(X_{\Box},Y_{\Box},I)$, $\mathscr{B}(X_{\Diamond},Y_{\Diamond},I)$, and $\mathscr{B}(X_{\uparrow},Y_{\downarrow},I)$, the corresponding sets of extents and intents are denoted analogously.

2.3. Definability by fuzzy Galois connections $\langle^{\uparrow},^{\downarrow}\rangle$

The following is our crucial observation. Recall that for a fuzzy set $A \in L^U$ and a truth degree $a \in L$, and a-cut of A is an ordinary subset ${}^aA$ of U defined by

$$ {}^aA = \{u \in U | A(u) \geqslant a\}. $$

In particular, 1-cut of A is an ordinary set ${}^1A = \{u \in U | A(u) = 1\}$. Furthermore, for a fuzzy relation I between X and Y, and a degree $\delta \in L$, a fuzzy relation $\delta \to I$ (δ-shift of I) between X and I is defined by

$$ (\delta \to I)(x,y) = \delta \to I(x,y). $$

Lemma 1. *For $A \in L^X$, $C \in 2^X$, $B \in L^Y$, $D \in 2^Y$, we have*

$$ C^* = {}^1(C^{\uparrow_{\delta\to I}}), \quad D^* = {}^1(D^{\downarrow_{\delta\to I}}), $$
$$ C^{\Box}(y) = C^{\uparrow_{\delta\to I}}, \quad B^{\Box} = {}^1(B^{\downarrow_{\delta\to I}}), $$
$$ A^{\Diamond} = {}^1(A^{\uparrow_{\delta\to I}}), \quad D^{\Diamond}(x) = D^{\downarrow_{\delta\to I}}, $$

where $^{\uparrow_{\delta\to I}} : L^X \to L^Y$ and $^{\downarrow_{\delta\to I}} : L^Y \to L^X$ are the operators induced by (1) and (2) by a fuzzy relation $\delta \to I$.

Proof. We prove only the case of $\langle^{\Diamond},^{\Diamond}\rangle$; the remaining cases are similar. Using $a \to \wedge_{j\in J} b_j = \wedge_{j\in J}(a \to b_j)$, $a \to (b \to c) = b \to (a \to c)$, and the fact that $a \leqslant b$ iff $a \to b = 1$, we have

$$ y \in A^{\Diamond} \quad \text{iff } \delta \leqslant \wedge_{x\in X}(A(x) \to I(x,y)) \quad \text{iff } 1 \leqslant \delta \to \wedge_{x\in X}(A(x) \to I(x,y)) = \wedge_{x\in X}(\delta \to (A(x) \to I(x,y))) $$
$$ = \wedge_{x\in X}(A(x) \to (\delta \to I)(x,y)) = A^{\uparrow_{\delta\to I}}(y), $$

proving $A^{\Diamond} = {}^1(A^{\uparrow_{\delta\to I}})$. Furthermore,

$$ D^{\Diamond}(x) = \delta \to \wedge_{y\in D} I(x,y) = \wedge_{y\in D}(\delta \to I(x,y)) = \wedge_{y\in Y}(D(y) \to (\delta \to I)(x,y)) = D^{\downarrow_{\delta\to I}}(x), $$

proving $D^{\Diamond} = D^{\downarrow_{\delta\to I}}$. $\Box$

Remark. Lemma 1 says that the pairs of operators $\langle^*,^*\rangle$, $\langle^{\Box},^{\Box}\rangle$, and $\langle^{\Diamond},^{\Diamond}\rangle$ are definable using the pair $\langle^{\uparrow},^{\downarrow}\rangle$.

2.4. Consequences of definability of $\langle^*,^*\rangle$, $\langle^{\Box},^{\Box}\rangle$, and $\langle^{\Diamond},^{\Diamond}\rangle$

Our point now is that taking Lemma 1 into account, one can, almost automatically, obtain results concerning the properties of $\langle^*,^*\rangle$, $\langle^{\Box},^{\Box}\rangle$, and $\langle^{\Diamond},^{\Diamond}\rangle$, and the corresponding concept lattices $\mathscr{B}(X_*, Y_*, I)$, $\mathscr{B}(X_{\Box}, Y_{\Box}, I)$, $\mathscr{B}(X_{\Diamond}, Y_{\Diamond}, I)$. Namely, the defining operators from Lemma 1 and the corresponding concept lattices have been studied in [5,7,16,8] and all we need to do is to apply the appropriate results from [5,7,16,8] together with Lemma 1. In what follows, we present the details.

Case of $\langle^*,^*\rangle$ and $\mathscr{B}(X_*, Y_*, I)$. The situation is particularly easy in this case. Namely, consider the δ-cut ${}^{\delta}I$ of I, i.e. ${}^{\delta}I = \{\langle x,y\rangle \in X \times Y \mid I(x,y) \geqslant \delta\}$. As an ordinary relation, ${}^{\delta}I$ induces an ordinary Galois connection $\langle^{\Uparrow_{\delta_I}},^{\Downarrow_{\delta_I}}\rangle$ between X and Y, i.e. for $C \in 2^X$ and $D \in 2^Y$ we have

$$ C^{\Uparrow_{\delta_I}} = \{y \in Y | \text{ for each } x \in C : \langle x,y\rangle \in {}^{\delta}I\}, $$
$$ D^{\Downarrow_{\delta_I}} = \{x \in X | \text{ for each } y \in D : \langle x,y\rangle \in {}^{\delta}I\}. $$

An easy observation shows that $C^{\Uparrow_{\delta_I}} = {}^1(C^{\uparrow_{\delta\to I}})$ and $D^{\Downarrow_{\delta_I}} = {}^1(D^{\downarrow_{\delta\to I}})$. Taking into account Lemma 1, we obtain

$$ C^* = C^{\Uparrow_{\delta_I}} \quad \text{and} \quad D^* = D^{\Downarrow_{\delta_I}}. $$

Furthermore, for each $C \in 2^X$, the set ${}^1C^{\uparrow_{\delta\to I}} = C^*$ is an intent of $\mathscr{B}(X_*, Y_*, I)$ and also an intent of $\mathscr{B}(X_{\Diamond}, Y_{\Diamond}, I)$ (due to Lemma 1 and the fact that $^{\Diamond}$ and $^{\Diamond}$ form a Galois connection which we will see next). Dually, for each $D \in 2^Y$, ${}^1D^{\downarrow_{\delta\to I}} = D^*$ is and extent of $\mathscr{B}(X_*, Y_*, I)$ and also an extent of $\mathscr{B}(X_{\Box}, Y_{\Box}, I)$.

Theorem 2. *Consider an arbitrary I, δ, and the corresponding $\langle {}^*, {}^* \rangle$ as described above.*

(1) $\langle {}^*, {}^* \rangle$ *forms a Galois connection between partially ordered sets* $\langle 2^X, \subseteq \rangle$ *and* $\langle 2^Y, \subseteq \rangle$ *which corresponds to an ordinary relation* ${}^\delta I$. *Therefore,* $\mathscr{B}(X_*, Y_*, I)$ *equals the ordinary concept lattice* $\mathscr{B}(X, Y, {}^\delta I)$.
(2) $\mathrm{Int}(X_*, Y_*, I)$ *is a subset of* $\mathrm{Int}(X_\diamond, Y_\diamond, I)$ *and* $\mathrm{Ext}(X_*, Y_*, I)$ *is a subset of* $\mathrm{Ext}(X_\Box, Y_\Box, I)$.

Remark

(1) The fact that $\langle {}^*, {}^* \rangle$ forms a Galois connection is proved in [10] in a direct way, i.e., without noticing that $\mathscr{B}(X_*, Y_*, I)$ is in fact an ordinary concept lattice. Part (2) of Theorem 2 is proved in [10] in a direct way, i.e. without the reference to known results.
(2) We omit details and just mention further ramifications of part (1) of Theorem 2. First, it gives a version of the Main theorem of concept lattices [11] for $\mathscr{B}(X_*, Y_*, I)$. In particular, one can see that an arbitrary complete lattice $\langle V, \leqslant \rangle$ is isomorphic to $\mathscr{B}(X_*, Y_*, I)$ iff there are mappings $\gamma X \to V$ and $\mu : Y \to V$ such that $\gamma(X)$ is $\vee$-dense in V, $\mu(Y)$ is $\wedge$-dense in V, and $\gamma(x) \leqslant \mu(y)$ iff $\delta \leqslant I(x, y)$. Second, it gives a way to efficiently compute $\mathscr{B}(X_*, Y_*, I)$ (using any algorithm for computing ordinary concept lattices).

Case of $\langle {}^\diamond, {}^\diamond \rangle$ and $\mathscr{B}(X_\diamond, Y_\diamond, I)$: It follows directly from Lemma 1 and [Theorem 2] [7] that $\mathscr{B}(X_\diamond, Y_\diamond, I)$ is a so-called "one-sided fuzzy concept lattice" with fuzzy extents and crisp intents which is induced by $\langle X, Y, \delta \to I \rangle$ with $\langle {}^\diamond, {}^\diamond \rangle$ being the corresponding Galois connection between $\langle L^X, \subseteq \rangle$ and $\langle 2^Y, \subseteq \rangle$. Note that one-sided concept lattices were introduced independently in [16,8] and, in a slightly different fashion, in [5]. Therefore, it follows immediately from [5, Corollary 1] and [7] (particularly from Theorem 2) that the set $\mathrm{Ext}(X_\diamond, Y_\diamond, I)$ of all extents of $\mathscr{B}(X_\diamond, Y_\diamond, I)$ is a subset of the set $\mathrm{Ext}(X, Y, \delta \to I)$ of all extents of $\mathscr{B}(X, Y, \delta \to I)$, and, moreover, that $\langle \mathrm{Ext}(X_\diamond, Y_\diamond, I), \subseteq \rangle$, as a complete lattice, is a $\wedge$-subsemilattice of the complete lattice $\langle \mathrm{Ext}(X, Y, \delta \to I), \subseteq \rangle$. Therefore, we have the following theorem.

Theorem 3. *Consider an arbitrary I, δ, and the corresponding $\langle {}^\diamond, {}^\diamond \rangle$ as described above.*

(1) $\langle {}^\diamond, {}^\diamond \rangle$ *forms a Galois connection between partially ordered sets* $\langle L^X, \subseteq \rangle$ *and* $\langle 2^Y, \subseteq \rangle$.
(2) $\langle \mathrm{Ext}(X_\diamond, Y_\diamond, I), \subseteq \rangle$, *is a complete lattice which is a $\wedge$-subsemilattice of the complete lattice* $\langle \mathrm{Ext}(X, Y, \delta \to I), \subseteq \rangle$.

Case of $\langle {}^\Box, {}^\Box \rangle$ and $\mathscr{B}(X_\Box, Y_\Box, I)$: This case is completely dual to that of $\langle {}^\diamond, {}^\diamond \rangle$ and we therefore just present the result.

Theorem 4. *Consider an arbitrary I, δ, and the corresponding $\langle {}^\Box, {}^\Box \rangle$ as described above.*

(1) $\langle {}^\Box, {}^\Box \rangle$ *forms a Galois connection between partially ordered sets* $\langle L^X, \subseteq \rangle$ *and* $\langle 2^Y, \subseteq \rangle$.
(2) $\langle \mathrm{Int}(X_\Box, Y_\Box, I), \subseteq \rangle$, *is a complete lattice which is a $\wedge$-subsemilattice of the complete lattice* $\langle \mathrm{Int}(X, Y, \delta \to I), \subseteq \rangle$.

Remark

(1) The fact that $\langle {}^\diamond, {}^\diamond \rangle$ and $\langle {}^\Box, {}^\Box \rangle$ form Galois connections is proved in [10] in a direct way. Parts (2) of Theorems 3 and 4 are stronger than the assertions of [10].
(2) As in case of Remark 2.4, parts (1) of of Theorems 3 and 4 have further ramifications (versions of Main theorem for $\mathscr{B}(X_\diamond, Y_\diamond, I)$ and $\mathscr{B}(X_\Box, Y_\Box, I)$, algorithms); details can be obtained by invoking the above relationships and results from [5,7]. Furthermore, using results from [5,7], one can obtain axiomatizations of the Galois connections $\langle {}^\diamond, {}^\diamond \rangle$ and $\langle {}^\Box, {}^\Box \rangle$, i.e. a list of sufficient and necessary properties of pairs $\langle {}^\diamond, {}^\diamond \rangle$ and $\langle {}^\Box, {}^\Box \rangle$ to be induced by some fuzzy relation I and a threshold δ using (7) and (8), and (5) and (6). Note that Theorems 3 and 4 do not present sufficient and necessary conditions; neither do the results in [10].

3. Conclusions

We showed that the new extent- and intent-forming operators from [10] can be defined in terms of the original operators from [1,3,17] and that several properties of the new operators can be obtained from the known properties of the original operators. Let us stress, however, that our note is primarily of technical importance: it enables us to use known results to study the new operators. The fact remains that the new operators provide a new parameterized way to obtain formal concepts from data with fuzzy attributes. This can be seen as an alternative to a parameterized approach via hedges [6].

Note that since the submission of the present paper, a new paper studying the relationships between the approach of [10] and the one of [6] appeared, see [4].

Note also that one may consider a natural generalization of the present approach using thresholds which consists in considering possibly different thresholds for different attributes. This generalization, suggested by a referee, corresponds to our proposal within the approach via hedges which consists in considering possibly different hedges for different attributes. A paper on this topic is in preparation.

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