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Fuzzy Sets and Systems 281 (2015) 61–72

FUZZY sets and systems

www.elsevier.com/locate/fss

Pavelka-style fuzzy logic in retrospect and prospect

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Received 3 February 2015; received in revised form 23 June 2015; accepted 9 July 2015

Available online 17 July 2015

Dedicated to Lotfi Zadeh

Abstract

We trace the origin and development of Pavelka-style fuzzy logic, discuss its significance and clarify some related misconceptions.

Keywords: Fuzzy logic; Pavelka-style logic; Vagueness; Uncertainty

1. Introduction

Shortly after Zadeh's seminal 1965 paper on fuzzy sets [41], two papers by Joseph Goguen appeared. Goguen's remarkable, visionary publications—the *L-fuzzy sets* of 1967 [19] and *The logic of inexact concepts* of 1968 [20]—led to a number of developments of various topics in fuzzy logic. Among the most important ones is the development of fuzzy logic as a logical calculus to which Goguen's papers provided an original, inspiring conception but also specific technical contributions. Goguen's conception served as an inspiration to Jan Pavelka who developed it in a series of papers [29–31] in the late 1970s. The resulting *Pavelka-style (fuzzy) logic* nowadays represents an important branch of fuzzy logic in the narrow sense. The 50th anniversary of the publication of Zadeh's seminal paper, which is considered the birth of fuzzy logic, provides an opportunity to reflect upon the origin, the development as well as the still prevailing misconceptions regarding Pavelka-style logic. A brief account of these is the subject of this paper.

2. Roots of Pavelka-style logic

2.1. Two sources

Pavelka-style logic has its roots in the two aforementioned Goguen's papers, particularly [20], and in Tarski's view of a deductive system as a consequence operator [35]. Thanks to the latter, which allows a very general view of a

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http://dx.doi.org/10.1016/j.fss.2015.07.007

logical calculus,[1] Pavelka-style logic represents a considerably broad logical framework subsuming diverse formal systems. Thanks to the former, to which I now turn, Pavelka-style logic naturally embodies the idea of approximate reasoning, in particular of what became known as a graded approach to truth.

2.2. Goguen's seminal contribution

Ever since their appearance, Goguen's papers [19,20] have been and continue to be—almost fifty years after their publication—an essential reading in fuzzy logic. This is so due to their clarity, a clean treatment of a broad spectrum of conceptual issues, as well as a reasonable technical depth. Here I restrict to the aspects directly related to Pavelka-style logic and refer to the upcoming book [2] for more information.

When regarded from a logical and mathematical viewpoint, Goguen's work represents a natural development of the initial ideas regarding fuzzy sets by Zadeh, of whom Goguen was a doctoral student at Berkeley. Its logical part differs substantially from the then recent contributions to many-valued logic in an important aspect. Namely, most of these contributions did not pay attention to the meaning of additional truth values. This was not true of some of the original contributions to many-valued logic, namely those by Łukasiewicz, Kleene, Bochvar and others who were guided by a more or less clear meaning of truth values employed in their logics, see [21,25] and [2]. However, in the other and later contributions to many-valued logic, interpretation of truth values has often been considered problematic or not not considered at all. Consequently, most of these contributions were in fact developing systems of many-valued logic which were, however mathematically sophisticated, somewhat sterile. Goguen, on the other hand, consistently interpreted truth values as degrees of truth and, in a sense, derived the novel conception of his "logic of inexact concepts" from this interpretation and from concrete problems in human reasoning. I consider this a remarkable and noteworthy moment in the development of many-valued logic and fuzzy logic in particular demonstrating a commonly encountered situation in mathematics and logic—a reasonable practical motivation guiding in a search for and eventually leading to a conceptually new and useful theory.

One concrete problem in human reasoning Goguen considered was the well-known sorites paradox, for which he refers to Max Black's treatment [7]. Black, who significantly contributed to the understanding of vagueness by his seminal contribution [6] in which he proposed predecessors of fuzzy sets, so called consistency profiles, may in this respect be considered an important source of inspiration for Pavelka-style fuzzy logic.

Leaving details aside, the crux of Goguen's resolution of the sorites paradox derives from the following observation [20, pp. 336, 371]:

> Just as propositions ... are no longer either 'true' or 'false', but can be intermediate, deductions are no longer 'valid' or 'invalid'. ... It seems to us that the present methods provide a framework for an 'inexact mathematics' in which we can apply approximately valid deductive procedures to approximately true hypotheses. The 'logic of inexact concepts' then helps assess the validity of the final conclusions.

In particular, Goguen examines in presence of a scale L of truth degrees the deduction rule of *modus ponens*, whose classical form reads "from φ and $\varphi \to \psi$ infer ψ" and is displayed as $\frac{\varphi, \varphi\to\psi}{\psi}$[2]:

> First, how is deduction used? We have a truth value $[P]$ and a truth value $[P \Rightarrow Q]$, and we want to estimate the truth value $[Q]$;

arriving thus at a more general rule

$$\frac{\varphi \text{ with degree } a, \varphi \to \psi \text{ with degree } c}{\psi \text{ with degree } a \otimes^{\cdot} c} \tag{1}$$

where $\otimes^{\cdot}$ is a truth function of a many-valued conjunction $\otimes$ adjoint to $\to$. One may observe that in the classical case, i.e. $L = \{0, 1\}$, Goguen's *modus ponens* may be identified with the ordinary one. The rule also has the intuitively appealing property that the result of "a long chain of only slightly unreliable deductions can be very un-

[1] See e.g. [39].

[2] He denotes formulas by P, Q, etc., implication by $\Rightarrow$, and the truth value of P by $[P]$. We denote formulas by $\varphi, \psi, \ldots$, logical connectives by $\to$ (implication), $\otimes$ (conjunction), ..., their truth functions by $\to^{\cdot}$, $\otimes^{\cdot}$, ..., and the truth value of formula φ by $\|\varphi\|$.

reliable" [20, p. 327]. Namely, suppose the validity of φ_0 is a and that of $\varphi_{i-1} \to \varphi_i$ is c_i for $i = 1, \ldots, n$. Then we may infer φ_1 to degree $a \otimes^\cdot c_1$ by a single application of (1) to φ_0 with a and $\varphi_0 \to \varphi_1$ with c_1; then also φ_2 to degree $a \otimes^\cdot c_1 \otimes^\cdot c_2$, and so on. After n steps, we obtain φ_n to degree $a \otimes^\cdot c_1 \otimes^\cdot \cdots \otimes^\cdot c_n$. With conjunctions $\otimes^\cdot$ such as the product, proposed by Goguen to resolve the sorites paradox, the degree $a \otimes^\cdot c_1 \otimes^\cdot \cdots \otimes^\cdot c_n$ may be much smaller than any of the degrees $a, c_1, \ldots, c_n$ of the assumptions. For instance, even if $a = 1$ and $c_i = 0.95$ for all i, the degree of the conclusion φ_{15} obtained after 15 inference steps is just $a \cdot c_1 \cdot \cdots \cdot c_{15} = 0.95^{15} \approx 0.46$.

Another contribution of Goguen relevant for our purpose is his introduction of complete (not necessarily commutative) residuated lattices [38] (which he called complete lattice ordered semigroups) as structures of truth degrees for fuzzy logic along with his logical justification of their algebraic properties.

3. Pavelka's *On fuzzy logic I–III*

Jan Pavelka (1948–2007)[3] developed the foundations of Pavelka-style logic during his PhD studies at Charles University in Prague and published them in [29–31]. According to Aleš Pultr, who served as Pavelka's advisor, he was a gifted hardworking student who strived for perfection in every aspect of his work. After the defense of his CSc. thesis[4] in 1978, he intended to work at the university but, except for a short period, there was no position for him available. He thus left the university, disappointed, to work as programmer in a company, which work he eventually quite enjoyed. According to recollections of Vilém Novák, in the 1980s Pavelka even regarded doing theoretical mathematics of little use. After he had left the university, he nevertheless continued to maintain contacts with academia and followed to certain extent further development in the field but did not himself contribute to it. After the Velvet Revolution in Czechoslovakia in 1989, he founded and successfully directed a private company specializing in information technology. At the same time, he had been teaching at Charles University, mainly practically oriented computer science courses, until his death in 2007.

3.1. Pavelka's contributions in brief

In the series [29–31], which according to the first paragraph of [29] "owes its inspiration to J.A. Goguen's comprehensive essay [20]", Pavelka worked out several notions and results at various levels of generality. In my experience, there is a widespread misunderstanding about what Pavelka actually did (see Section 4.2). Therefore, I now present an overview of Pavelka's contributions with comments regarding certain issues. The contributions in [29–31] may naturally be split in two parts. The first is the content of [29] and consists in developing what may be called an abstract fuzzy logic. The second is the content of [30,31] and consists in developing propositional logics as particular instances of the abstract fuzzy logic.

Pavelka's abstract fuzzy logic, to which the term "Pavelka-style logic" properly refers, encompasses a rich variety of interesting logical calculi. These include the aforementioned (truth-functional) propositional logics studied by Pavelka himself but also many other calculi developed later. Importantly, they also include calculi which are not truth functional (see below). To identify Pavelka-style logic and Pavelka's contribution with the particular propositional fuzzy logics of [30,31], which is frequently happening, is thus a gross misconception.

3.2. Part I: abstract fuzzy logic

An *abstract fuzzy logic*[5] may be thought of as a tuple $\langle \mathcal{F}, \mathbf{L}, \mathcal{S}, A, \mathcal{R} \rangle$ consisting of an (abstract) set $\mathcal{F}$ of *formulas*; a complete lattice $\mathbf{L} = \langle L, \leq, \cdots \rangle$ (possibly with additional operations) of *truth degrees*; an *L-semantics* $\mathcal{S}$, which is

[3] The historical information in this paragraph is based on recollections of Pavelka's wife Emilia Pavelková, Professor A. Pultr, Professor V. Novák, and Pavelka's colleagues. For more information see [2].

[4] "Reziduované svazy a logika s neurčitostí" [Residuated Lattices and Logic with Uncertainty]; The Czechoslovak equivalent to PhD was "candidate of sciences", abbreviated "CSc."

[5] This name is used by Hájek [23] and Gerla [13]. Pavelka did not have any name for it. Our terminology and notation slightly differs from Pavelka's.

an arbitrary set $\mathcal{S} \subseteq L^{\mathcal{F}}$ of fuzzy sets of formulas; a fuzzy set $A \in L^{\mathcal{F}}$ of *axioms*; and a set $\mathcal{R}$ of *deduction rules* explained below.[6]

The formulas φ in $\mathcal{F}$ may be built up inductively from atomic ones as usual but need not have any inner structure and may simply represent certain statements in an *ad hoc* way. The elements a in L, which may but need not be numbers and include the boundary 0 and 1, are primarily interpreted as truth degrees and **L** plays the role of a structure (algebra) of truth degrees. The fuzzy sets E in $\mathcal{S}$ play the role of truth evaluations (or semantic structures in which formulas assume truth degrees): for $\varphi \in \mathcal{F}$ and $E \in \mathcal{S}$, we denote $E(\varphi) \in L$ also by $\|\varphi\|_E$ and call it the truth degree of φ in E. The degree $A(\varphi) \in L$ is interpreted as the degree to which the formula φ is considered as axiom. The deduction rules are inspired by Goguen's (1): each (n-ary) deduction rule $R \in \mathcal{R}$ is a pair $R = \langle R_{\text{syn}}, R_{\text{sem}}\rangle$ consisting of a partial function $R_{\text{syn}} : \mathcal{F}^n \to \mathcal{F}$ (syntactic part) and a function $R_{\text{sem}} : L^n \to L$ (semantic part)[7] and is visualized as

$$\frac{\langle \varphi_1, a_1\rangle, \ldots, \langle \varphi_n, a_n\rangle}{\langle \varphi, a\rangle}, \quad \text{with } \varphi = R_{\text{syn}}(\varphi_1, \ldots, \varphi_n), \text{ and } a = R_{\text{sem}}(a_1, \ldots, a_n). \tag{2}$$

The intended meaning is: from the validity of φ_i to degree (at least) a_i, $i = 1, \ldots, n$, infer that φ is valid to degree (at least) a. Thus, Goguen's *modus ponens* would be depicted as $\frac{\langle \varphi, a\rangle, \langle \varphi \to \psi, c\rangle}{\langle \psi, a \otimes^{\cdot} c\rangle}$.

Example 1. To see that abstract fuzzy logics subsume very diverse kinds of semantics, consider two examples.

(a) Łukasiewicz logic. Let $\mathcal{F}$ be the set of all formulas of the infinitely-valued Łukasiewicz logic, **L** be [0, 1] with its natural order and define the L-semantics as

$$\mathcal{S} = \{E \in [0,1]^{\mathcal{F}} \mid \text{ for some evaluation } e : E(\varphi) = \|\varphi\|_e \text{ for each } \varphi \in \mathcal{F}\}.$$

In this case, $E \in \mathcal{S}$ are just the truth evaluations induced by evaluations of propositional symbols and $E(\varphi)$ is the truth degree of φ.

(b) Probabilistic logic. Let $\Omega \neq \emptyset$ be a finite set with $\omega \in \Omega$ called elementary events, **L** be [0, 1] again, and $\mathcal{F} = 2^{\Omega}$ be the set of events over Ω. Then

$$\mathcal{S} = \{E \in [0,1]^{\mathcal{F}} \mid \text{for some probability distribution } p \text{ on } \Omega : E(\varphi) = \sum_{\omega \in \varphi} p(\omega)\}$$

is an L-semantics. In this case, $E \in \mathcal{S}$ are just the probability measures on Ω and $E(\varphi)$ is the probability of φ.

The semantics in example (a) is well known and we now see how it is looked at in Pavelka's framework. It is *truth functional* in that for every connective c, there exists a function $c^{\cdot}$ (the truth function of this connective) such that the truth degree of a compound formula (which results by application of c to other, constituent formulas) is obtained by applying $c^{\cdot}$ to the truth degrees of the constituent formulas. Thus, for conjunction, $\|\varphi \otimes \psi\|_E = \|\varphi\|_E \otimes^{\cdot} \|\psi\|_E$, where $a \otimes^{\cdot} b = \max(0, a + b - 1)$, and the same for the other connectives.

Example (b), on the other hand, represents a non-truth-functional semantics. In particular, the set $\mathcal{F}$ of formulas may be defined as follows: each $\omega \in \Omega$ is a formula; if $\varphi, \psi \in \mathcal{F}$ then $\varphi \cap \psi \in \mathcal{F}$ and $\Omega - \varphi \in \mathcal{F}$. As is well known,[8] there is no function $\cap^{\cdot}$ such that $\|\varphi \cap \psi\|_E = \|\varphi\|_E \cap^{\cdot} \|\psi\|_E$ for every $\varphi, \psi \in \mathcal{F}$, i.e. the probability of the intersection of two events is a function of the probabilities of these events. We shall see below that several fuzzy logics have been developed, in which degrees of truth are interpreted as degrees of uncertainty and which are not truth functional. These are very different from the mainstream, truth-functional logics and are not widely known.

In classical logic, a theory is a set of formulas, which may be considered as assumptions from which one may derive further formulas. In fuzzy logic, it is natural to suppose that each assumption, φ, comes with its truth degree, $T(\varphi)$, prescribed by the theory T. Hence, according to Pavelka, a *theory* T is an arbitrary fuzzy set of formulas, i.e. $T \in L^{\mathcal{F}}$. Naturally then, a *model* of a theory T is any fuzzy set $E \in \mathcal{S}$ for which $T(\varphi) \leq E(\varphi)$ for any $\varphi \in \mathcal{F}$, i.e. the degree to which φ is true in E is at least as high as the degree prescribed by T.

6 A may be considered as a collection of nullary deduction rules.

7 Pavelka moreover assumes that R_{sem} commutes with suprema and later considers also just the isotony of R_{sem}.

8 In the context of logical treatment of probability, this was likely first observed by S. Mazurkiewicz, "Zur Axiomatik der Wahrscheinlichkeitslehre," *Comptes Rendus des Séances de la Societé des Sciences et des Lettres de Varsovie, Cl. iii*, 25 (1932): 1–4.

Example 2. For instance, the theory

$$T = \{^{young(John)}/0.8, {}^{baby(Joe)}/0.9, \ldots, {}^{baby(x)\to young(x)}/1\}$$

asserts that *John* is young to degree (at least) 0.8 and so on.

Returning to Example 1(b), let $\Omega = \{1, \ldots, 6\}$ represent tossing a die. With a fair die, we have $p(1) = \cdots = p(6) = \frac{1}{6}$ for the corresponding probability distribution p. However, for a biased die we might have $p_b(1) = \cdots = p_b(4) = \frac{1}{6}$, $p_b(5) = \frac{1}{12}$, $p_b(6) = \frac{3}{12}$ (6 is more likely than 5). The theory $T = \{^{\frac{1}{12}}/\{1\}, \ldots, {}^{\frac{1}{12}}/\{6\}\}$ might represent the requirement that the probability of each result i be at least $\frac{1}{12}$, i.e. the die cannot be biased too much. Clearly, the evaluation E corresponding to the above distribution p_b is a model of T, while E corresponding to a fake die containing only even numbers is not.

Denoting the models of T by $\mathrm{Mod}(T)$, the *degree* $\|\varphi\|_T$ *to which* φ *semantically follows from* T is defined by

$$\|\varphi\|_T = \bigwedge_{E \in \mathrm{Mod}(T)} \|\varphi\|_E,$$

i.e. the infimum of degrees to which φ is true in models of T which is naturally interpreted as the degree of the statement "φ is true in every model of T".

The degree of entailment, as a semantic notion, has its counterpart in the syntactic notion of degree of provability, which naturally implements the idea of deriving partially true conclusions from partially true assumptions. An (*L-weighted*) *proof* from T is a finite sequence $\langle\varphi_1, a_1\rangle, \ldots, \langle\varphi_n, a_n\rangle$ of (*L-weighted*) *formulas*, i.e. pairs $\langle\varphi_i, a_i\rangle$ of $\varphi_i \in \mathcal{F}$ and $a_i \in L$, such that for each i, $a_i = A(\varphi_i)$ (axiom) or $a_i = T(\varphi_i)$ (assumption) or $\langle\varphi_i, a_i\rangle$ is obtained from some $\langle\varphi_j, a_j\rangle$s, $j < i$, by some rule $R \in \mathcal{R}$, i.e. $\varphi_i = R_{\mathrm{syn}}(\ldots, \varphi_j, \ldots)$ and $a_i = R_{\mathrm{sem}}(\ldots, a_j, \ldots)$ for some indices $j < i$. The *degree* $|\varphi|_T$ *of provability* of a formula $\varphi \in \mathcal{F}$ from a theory $T \in L^{\mathcal{F}}$ is defined as

$$|\varphi|_T = \bigvee\{a_n \mid \langle\varphi_1, a_1\rangle, \ldots, \langle\varphi_n, a_n\rangle \text{ is a proof from } T\}. \tag{3}$$

An abstract fuzzy logic $\langle\mathcal{F}, \mathbf{L}, \mathcal{S}, \mathcal{R}\rangle$ is (*Pavelka-style*) *complete* if

$$\text{for each theory } T \in L^{\mathcal{F}} \text{ and formula } \varphi \in \mathcal{F} \text{ we have} \quad |\varphi|_T = \|\varphi\|_T, \tag{4}$$

i.e. degrees of semantic entailment equal degrees of provability.

Observe that in the bivalent case, i.e. $L = \{0, 1\}$, the notions of abstract fuzzy logic, such as that of theory, model, semantic consequence, deduction rule, and proof, virtually coincide with (abstractly conceived) notions of classical logic. In particular, Pavelka-style completeness, (4), becomes classical completeness and says that φ is provable from T if and only if φ semantically follows from T.

3.3. Parts II and III: propositional logics as particular abstract fuzzy logics

In these parts, Pavelka examines—as particular examples of abstract fuzzy logics—certain propositional logics, both finitely and infinitely valued, in which—inspired again by Goguen—$\mathbf{L}$ is assumed to be a complete residuated lattice $\langle L, \dot\wedge, \dot\vee, \dot\otimes, \dot\to, 0, 1\rangle$ (equipped possibly with further operations), whose operations are the truth functions of the corresponding connectives, denoted $\wedge, \ldots, \to$. The language of these logics contains the symbols of connectives, propositional symbols $p, q, \ldots$, auxiliary symbols, and in addition a symbol $\bar{a}$—the *truth constant* of a—for every truth degree $a \in L$.[9] The logic has the usual truth-functional semantics $\mathcal{S}$, i.e. the evaluations $E \in \mathcal{S}$ satisfy $\|\varphi \wedge \psi\|_E = \|\varphi\|_E \dot\wedge \|\psi\|_E$ and the same for the other connectives, plus $\|\bar{a}\|_E = a$.

Pavelka solved the problem of axiomatizability of such logics, L, in case $\mathbf{L}$ is an arbitrary residuated lattice with L being a finite chain or the real unit interval [0, 1].[10] His main results may be summarized as follows:

Theorem 1. *(a) If L is a finite chain then there exists a fuzzy set A of axioms and a set $\mathcal{R}$ of deduction rules such that the logic L is Pavelka-style complete.*

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9 This may make the language uncountable and hence not recursively enumerable (e.g. if L is the real unit interval [0, 1]). Pavelka does not pay attention to recursive aspects and neither do I in this paper. The reader is referred to [13], the more recent [5,14], and also to [23,28].

10 In fact, Pavelka considered so-called enriched residuated lattices, which may additionally contain further operations.

(b) If $L = [0, 1]$ *then there exist* A *and* $\mathcal{R}$ *such that* L *is Pavelka-style complete if and only if* $\mathbf{L}$ *is the standard Łukasiewicz algebra* $[0, 1]_Ł$ *or its isomorphic copy.*

In fact, Pavelka provided the axioms and deduction rules which demonstrate the axiomatizability in (a) and (b). For the non-axiomatizability in (b), he presented his now well-known argument showing that if the truth function of implication $\rightarrow^{\cdot}$ is not continuous, then the logic may not be Pavelka-style complete and recalled his earlier joint result [26] according to which any residuated lattice on [0, 1] with a continuous $\rightarrow^{\cdot}$ is an isomorphic copy of the standard Łukasiewicz algebra.

4. Significance, misconceptions, and developments regarding Pavelka-style logic

I now take a closer look at the significance, prevailing misconceptions and further development of Pavelka-style logics, and pay particular attention to Hájek's rational Pavelka logic and Gerla's contributions to abstract fuzzy logics.

4.1. Significance

Pavelka's contribution [29–31] is one of the most important contributions to fuzzy logic in the narrow sense (or, mathematical fuzzy logic). It is one the earliest contributions to fuzzy logic inspired by Zadeh's idea of fuzzy sets [41] with a clear logical content—the first is Goguen's [20] and partly already his [19], the other include Lee and C.L. Chang's [18], Lee's [17], and the contributions of Gaines, e.g. [11,12], and Giles [15].

From a broader outlook of many-valued logic, it is interesting to note that Pavelka was aware of Łukasiewicz's *Selected Works* [24] containing several contributions to Łukasiewicz logic and of Rosser and Turquette's book [34] on axiomatization of finitely-valued logics. He also knew the algebraic method of proving completeness described in Rasiowa and Sikorski's [32], which he indeed utilized. Nevertheless, he was probably not aware of the simple axiomatization by four axioms of the infinitely-valued Łukasiewicz logic (when conjectured by Łukasiewicz, it had five axioms; one had later been proved redundant). This axiomatization, whose first proof was worked out by Rose and Rosser [33], is presented already in Łukasiewicz and Tarski's *Investigations into the sentential calculus*, which appears in [24] to which Pavelka refers. Had Pavelka been aware of it, his axiomatization (justifying Theorem 1(b)) could have been much simpler than his actual axiomatization with over thirty axioms.

Nevertheless, Pavelka's conception of deriving partly true propositions from other partly true propositions using deduction rules with syntactic and semantic part, his grasp and examination of the notion of degree of semantic entailment and its syntactic counterpart, the degree of provability,—which were crucially inspired by Goguen's *Logic of inexact concepts*—clearly represent a highly original contribution to many-valued logic.

In my view, the key factor contributing to the emergence of these new logical concepts in Goguens's and Pavelka's works was the existence of a meaningful, intuitively appealing interpretation of truth values as degrees of truth of propositions that involve vague predicates. Such interpretation has convincingly been proposed in Zadeh's [41] and other papers. Zadeh's work is thus of crucial importance in this regard, even though the ideas essentially appeared already with the consistency profiles in Black's analysis of vagueness [6]. In a sense, this interpretation made possible a logical analysis of the various perplexities involving vague notions, such as the sorites paradox, which eventually lead to the notion of deduction to a degree. Note in this context that a meaningful interpretation of truth values in most many-valued calculi developed up to late 1960s was either missing and the calculi have thus been developed as purely formal, or the interpretation existed but did not naturally lead to the notion of degree of entailment and related notions (see the upcoming [2]). This documents that an interpretation, connecting a purely formal calculus of logic with human reasoning, may have an important impact on development of logical notions and logical agenda themselves. In the realm of fuzzy logic, the first examples—followed later by many others—are represented by Goguen's and Pavelka's works.

As mentioned above, Pavelka himself did not have a name for what is now called the abstract fuzzy logic. Apart from some technical examples of it, his propositional logics were the only ones he developed. This indicates that he invented the abstract fuzzy logic primarily as a framework in which he was developing his propositional logics. Importantly, however, many other examples of abstract fuzzy logics have been examined later on, including probabilistic logics which are not truth functional and hence much less conventional than Pavelka's own (truth functional) propositional fuzzy logics. As far as one can tell from Pavelka's publications, he was not aware of these “unintended consequences” of the high level generality of his framework. Nevertheless, the rich variety of particular examples certainly emphasizes the practical significance of Pavelka’s notion of abstract fuzzy logic.

4.2. Misconceptions

Pavelka’s seminal contribution is frequently cited in the literature[11] and is being mentioned in monographs on fuzzy logic. Nevertheless, in my experience, drawn from publications, talks and personal communication, a widespread misconception exists about Pavelka’s contributions.

First, Pavelka’s notion and results regarding abstract fuzzy logic are not recognized. Instead, the term “Pavelka-style logic” (or similar terms) is used to refer solely to the propositional fuzzy logics developed by Pavelka or possibly to other propositional or predicate fuzzy logics developed in the spirit of Pavelka’s propositional fuzzy logics, without realizing that, as explained above, these logics are just particular instances—even though very important ones—of the notion of abstract fuzzy logic.

Second, even if the notion of abstract fuzzy logic is recognized, it is believed that Pavelka’s propositional fuzzy logics (and similar logics developed later) are the only existing solid examples of abstract fuzzy logics. Therefore, it is not realized that many other abstract fuzzy logics have been developed and that the notion of abstract fuzzy logic represents a fruitful framework within which one may develop a variety of deductive systems for reasoning about graded truth and uncertainty, including both truth-functional and non-truth-functional logics. Various examples in this regard are presented below, in particular in Section 4.5.

Third, Pavelka’s propositional infinitely-valued Łukasiewicz logic is erroneously identified with Hájek’s Rational Pavelka logic. This issue is addressed in Section 4.4

Fourth, Pavelka’s negative result on axiomatizability of other than the Łukasiewicz propositional fuzzy logics (Theorem 1(b)) is interpreted as implying that whenever one desires to have a Pavelka-style completeness in Pavelka-style logics, one has to restrict to the Łukasiewicz operations. In fact, however, several Pavelka-style complete logics exist which are based on different operations, e.g. logics with restricted kind of formulas mentioned in Section 4.3.

4.3. Further developments in brief

In [31, p. 463], Pavelka mentions that he was able to extend his results on propositional fuzzy logics to the predicate case but the extension has never been published. The predicate case has been thoroughly studied by Novák, who finally presented it in [27] and made it a subject of many other papers. Instead of “Pavelka-style (fuzzy) logic”, Novák prefers the term “fuzzy logic with evaluated syntax”; in previous works, such as the book [28], he used “fuzzy logic with graded syntax”.[12] A typical feature of these papers is attention to natural language and extralogical motivations, which is an important contribution by itself. The reader is recommended the book [28]. Various aspects of Pavelka-style propositional and predicate fuzzy logics have also been studied by Ying, see e.g. [40], and Turunen, see e.g. [36].

Pavelka’s abstract fuzzy logic has soon became subject of papers by G. Gerla. Due to their particular importance in our paper, these works are examined in Section 4.5. Let us also note in this context that due to the generality of the notion of abstract fuzzy logic, the works on fuzzy closure operators are directly relevant to it, see e.g. the upcoming book [2].

Pavelka’s propositional fuzzy logics became an inspiration for studying fuzzy logics with truth constants. This topic started with Hájek’s Rational Pavelka logic and is the subject of Section 4.4.

We also mention the fuzzy equational and implicational logics, started in [1], see [3], and the fuzzy logic programming system [37], which may be thought of as fragments of Pavelka-style predicate fuzzy logics, and are Pavelka-style complete for general classes of structures of truth degrees. Worth mentioning is a logic of dependencies in data with fuzzy attributes, which represents a different kind of abstract fuzzy logic inspired by reasoning about data (see e.g. the recent [4]).

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[11] As of January 27, 2015, Google Scholar shows 796 citations of part I of [29]. In comparison, it shows 1001 citations of Goguen’s [20] which is generally considered as highly influential in fuzzy logic.

[12] The terminology differs correspondingly. In particular, Novák uses “evaluated proof”, “evaluated formula”, “fuzzy theory”, and “generalized completeness” of the above notion of weighted proof, weighted formula, theory, and Pavelka-style completeness.

4.4. Hájek's rational Pavelka logic

In one of his first papers on fuzzy logic [22], Hájek—who was reviewer of Pavelka's thesis—revisits Pavelka's infinitely-valued Łukasiewicz logic and presents a considerably simpler modification of it with a Pavelka-style completeness. In particular, he considers the truth constants for rationals in $[0, 1]$ and rationally-valued theories only (making the language countable and ready for examination of recursive properties), a much simpler set of axioms including to degree 1 the four axioms mentioned in Section 4.1 and further axioms regarding truth degrees (formula $\bar{a}$ to degree a for each rational $a \in [0, 1]$, and two schemes of what he calls bookkeeping axioms which relate truth degrees to connectives), but still considers truth evaluations by reals in $[0, 1]$. Hájek calls the resulting logic RPL—the Rational Pavelka logic.[13] This RPL is still a kind of abstract fuzzy logic with the notion of degree of provability defined as by Pavelka.

He later presented RPL in his [23]. This RPL, however, is conceptually different from the RPL of [22]. Namely, it is not an abstract fuzzy logic but rather an expansion of the ordinary Łukasiewicz infinitely-valued logic by truth constants for rationals with extra axioms regarding truth degrees (essentially same as those described above). In this logic, the genuine notions of abstract fuzzy logic, such as that of degree of provability, are "simulated" by the ordinary notions. The "simulation" is possible due to the fact that in Łukasiewicz logic, if an evaluation e satisfies $\|\bar{a}\|_e = a$—which Hájek assumes—then a formula φ is true to degree at least a if and only if the formula $\bar{a} \to \varphi$ is fully true, i.e. true to degree 1:

$$a \le \|\varphi\|_e \text{ iff } \|a \to \varphi\|_e = 1. \tag{5}$$

In particular, Hájek represents Pavelka's weighted formulas $\langle \varphi, a \rangle$ by ordinary formulas $\bar{a} \to \varphi$, which makes it possible to represent Pavelka's rational-valued theories T as ordinary theories $\{\overline{T(\varphi)} \to \varphi \mid \varphi \text{ a formula}\}$, considers Pavelka's deduction rules as ordinary deduction rules, e.g.

$$\frac{\langle \varphi, a \rangle, \langle \varphi \to \psi, c \rangle}{\langle \psi, a \otimes^{\cdot} c \rangle} \quad \text{as} \quad \frac{\bar{a} \to \varphi, \bar{a} \to (\varphi \to \psi)}{\overline{a \otimes^{\cdot} b} \to \psi},$$

proves that they are derivable in RPL, and defines the degree of provability of φ from a theory (i.e. a set of formulas) T by (cf. (3))

$$|\varphi|_T^{\mathrm{RPL}} = \bigvee \{\text{rational } a \in [0, 1] \mid \bar{a} \to \varphi \text{ is ordinarily provable from } T \text{ in RPL}\} \tag{6}$$

With a naturally defined (essentially as by Pavelka) degree of semantic entailment, $\|\varphi\|_T^{\mathrm{RPL}} = \bigwedge\{\|\varphi\|_e \mid e \text{ a model of } T\}$ (being a model here means $\|\varphi\|_e = 1$ for each $\varphi \in T$), Hájek then proves his Pavelka-style completeness, i.e. $|\varphi|_T^{\mathrm{RPL}} = \|\varphi\|_T^{\mathrm{RPL}}$ for every theory. Hájek's approach stimulated a stream of studies of (ordinary-style) fuzzy logics with truth constants in language and I refer to [10] for an overview of these.

The misconception mentioned in Section 4.2 consists in identifying Pavelka's propositional Łukasiewicz logic with the RPL as presented in [23] and thus also in believing that Pavelka defined the notion of degree of provability by (6).

Worth noting is also the fact that Hájek's "simulation" is possible thanks to the particular nature of the formulas and the logic involved, namely due to (5), and is not universally applicable—many important abstract fuzzy logics do not admit this kind of simulation.

4.5. Gerla's contributions and non-truth functional fuzzy logics of uncertainty

A number of important contributions to abstract fuzzy logic is due to G. Gerla, who embarked on its systematic study in the 1980s and published a number of papers, often coauthored with L. Biacino. A comprehensive account of these contributions is offered by his book [13], but several important topics are dealt with in his more recent papers, such as [14].

Gerla follows Pavelka's conception and studies fuzzy closure operators as consequence operators of fuzzy logics. This way, he obtains results for general classes of fuzzy logics as well as particular logics, including truth-functional

[13] He uses "Rational-valued Pavelka's logic", to be precise. Later on, he switched to "Rational Pavelka logic."

and non-truth-functional ones. He addresses a variety of topics including particular properties of fuzzy closure operators characteristic of consequence operators (such as continuity or compactness), relationship to ordinary consequence operators including various extensions of ordinary closure operators to fuzzy closure operators, so-called Hilbert systems which arise from Pavelka-style axioms and deduction rules, characterization of axiomatizability of fuzzy logics via properties of their consequence operators, and a variety of further ones.

To demonstrate that Pavelka's abstract fuzzy logics include a number of non-traditional logics, I now turn to Gerla's probabilistic logics. The intended formulas in these logics are those of the classical propositional logics, to which one assigns various kinds to degrees of belief in [0, 1] in such a way that tautologies are assigned 1 and that two classically equivalent formulas are always assigned the same degree. Correspondingly, Gerla assumes that the set $\mathcal{F}$ of formulas is a Lindenbaum algebra of classical logic and, still more generally, a Boolean algebra $\mathbf{B} = \langle B, \wedge, \vee, \neg, \bar{0}, \bar{1}\rangle$. Beside other such logics, he considers the *logic of superadditive measures*, L_{sa}, *logic of upper-lower probabilities*, L_{ul}, *logic of envelopes* (or *probability logic*), L_{p}, and the *logic of necessities*, L_{n}, which is closely related to the possibilistic logic discussed in the next section. For each of these logics, which correspond to important types of uncertainty, he defines its semantics, axioms and deduction rules and proves a Pavelka-style completeness.

For instance, the semantics $\mathcal{S}_{\mathrm{sa}}$ of L_{sa} consists of all constant-sum superadditive measures on $\mathbf{B}$, i.e. mappings $p : B \to [0, 1]$ satisfying $p(\bar{1}) = 1$, $p(\varphi \vee \phi) \geq p(\varphi) + p(\psi)$ provided $\varphi \wedge \psi = \bar{0}$, and $p(\varphi) + p(\neg\varphi) = 1$. L_{sa} has $\bar{1}$ to degree 1 as the only axiom and two deduction rules,

$$\frac{\langle\varphi, a\rangle, \langle\psi, b\rangle}{\langle\varphi \vee \psi, a \oplus b\rangle} \quad \text{and} \quad \frac{\langle\varphi, a\rangle, \langle\psi, b\rangle}{\langle\bar{0}, c(a, b)\rangle},$$

which are both defined for $\varphi \wedge \psi = \bar{0}$ and for which $\oplus$ is the truth function of Łukasiewicz disjunction, $a \oplus b = \min(1, a + b)$, and $c(a, b) = 1$ for $a + b > 1$ and $= 0$ for $a + b \leq 1$. Gerla is then able to prove Pavelka-style completeness of L_{sa}. For the other logics, I only present their semantics. For L_{ul}, the semantics $\mathcal{S}_{\mathrm{ul}}$ is the set of all upper-lower probabilities on $\mathbf{B}$, i.e. superadditive measures p on $\mathbf{B}$ whose dual function $p^{\perp}(\alpha) = 1 - p(\neg\alpha)$ satisfies subadditivity, i.e. $p^{\perp}(\varphi \vee \psi) \leq p^{\perp}(\varphi) \vee p^{\perp}(\psi)$. For L_{p}, the semantics $\mathcal{S}_{\mathrm{p}}$ is the set of all finitely additive probabilities on $\mathbf{B}$. Finally, the semantics $\mathcal{S}_{\mathrm{n}}$ of L_{n} is the set of all necessity measures on $\mathbf{B}$.

Interestingly, these logics form a natural hierarchy. Let $\mathcal{T}_*$ denote the set of all consistent closed theories of L_*, i.e. theories $T : \mathbf{B} \to [0, 1]$ which satisfy $T(\varphi) < 1$ for some φ and are closed w.r.t. deduction rules, i.e. $R_{\mathrm{sem}}(T(\varphi_1), \ldots, T(\varphi_n)) \leq T(R_{\mathrm{syn}}(\varphi_1, \ldots, \varphi_n))$ for every rule R of L_*. Then $\mathcal{T}_{\mathrm{sa}}$, $\mathcal{T}_{\mathrm{ul}}$, $\mathcal{T}_{\mathrm{p}}$, and $\mathcal{T}_{\mathrm{n}}$ are the sets of all superadditive measures, upper-lower probabilities, lower envelopes, and necessity measures on $\mathbf{B}$, and are related by

$$\mathcal{T}_{\mathrm{n}} \subseteq \mathcal{T}_{\mathrm{p}} \subseteq \mathcal{T}_{\mathrm{ul}} \subseteq \mathcal{T}_{\mathrm{sa}}.$$

5. Further remarks and prospects

As we have seen, abstract fuzzy logics represents a general framework which subsumes many diverse logics. In spite of its generality, its notions are natural and intuitively easy to understand. Its conception is directly linked to the idea of deriving approximately true, or partly uncertain, statements from other approximately true or uncertain statements, which frequently appears in many works discussing imprecision, inexactness and the applicability of classical logic and mathematics in this regard. Pavelka's notion of abstract fuzzy logic may be regarded as a natural framework for this idea.

It may be illustrative at this point to recall Zadeh's influential 1975 paper [42] in which he also discusses the intuitive notion of a fuzzy theorem as an approximately true statement that may be inferred from axioms by approximate reasoning. In particular, Zadeh presents an example of such theorem from plane geometry which concerns approximate triangles. A closer inspection reveals that Zadeh's ideas, which he develops more or less intuitively using a kind of qualitative reasoning, may naturally be approached via Pavelka's conception and offer an interesting research problem.

Pavelka's abstract fuzzy logic may be regarded as a practical simple tool. All one is required to obtain a reasonable logic is to supply the following. First, formulas which are arbitrary elements (e.g. strings or sets) representing one's desired statements. Second, semantic structures, i.e. representations of reality about which one makes the statements, with a rule assigning to the formulas their degrees of truth (or uncertainty) in each structure. Pavelka's conception than yields the notion of theory and entailment and one may attempt to axiomatize this entailment, i.e. to devise appropriate axioms and deduction rules. In this perspective and in view of the existing contributions, it may even be argued that the notion of abstract fuzzy logic, developed in part I and presumably considered as auxiliary by Pavelka, turned out to be of greater importance than the particular propositional fuzzy logics he developed in parts II and III.

The situation with Pavelka-style logic is in my view well described by saying that this kind of logic is a visible branch of fuzzy logic but, in spite of the many existing contributions, it is actually more visible than properly understood. As a concrete example in this regard I point to the comments in [9, p. 13] comparing some aspects of possibilistic logic and Pavelka-style logic to point out the differences between the two. The remarks regarding Pavelka-style logics are, by and large, misleading and are not based on a proper understanding of Pavelka's conception. The differences pointed out are not correct and the authors seem not to be aware of the fact that possibilistic logic naturally fits Pavelka's framework and may in fact be regarded as a particular abstract fuzzy logic, as shown below.[14]

Recall first the notions of the basic version of the possibilistic logic as described in [8]. Consider a classical propositional logic with its formulas denoted φ and the like. Let W be a set of all classical evaluations of the given propositional logic (W is conceived as a set of possible worlds). Each normalized possibility distribution π on W, i.e. a mapping $\pi : W \to [0,1]$ such that $\pi(w) = 1$ for some $w \in W$, induces mappings Π_π and N_π assigning to each formula φ the number $\Pi_\pi = \{\pi(w) \mid \|\varphi\|_w = 1\}$ (i.e. the possibility measure induced by π of the set of all evaluations in W which make φ true) and $N_\pi(\varphi) = 1 - \Pi_\pi(\neg\varphi)$ (necessity). A possibilistic formula is conceived as a pair $\langle\varphi, a\rangle$ where φ is a classical propositional formula and $a \in (0,1]$. Now, π satisfies $\langle\varphi, a\rangle$, in symbols $\pi \models \langle\varphi, a\rangle$, if $N_\pi(\varphi) \geq a$; $\langle\varphi, a\rangle$ semantically follows from a set T of possibilistic formulas (called a knowledge base), in symbols $T \models \langle\varphi, a\rangle$, if every normalized π satisfying each possibilistic formula in T satisfies $\langle\varphi, a\rangle$. In possibilistic logic, one is explicitly interested in the degree

$$\mathrm{Val}(\varphi, T) = \bigvee\{a \in (0,1] \mid T \models \langle\varphi, a\rangle\}.$$

The axiomatization of the present possibilistic logic, due to [16],[15] is given by its axioms, which are possibilistic formulas $\langle\varphi_1, 1\rangle, \ldots, \langle\varphi_k, 1\rangle$ with $\varphi_1, \ldots, \varphi_k$ being an arbitrary set of complete axioms of classical logic, and by two deduction rules,

$$\frac{\langle\varphi, a\rangle, \langle\varphi\to\psi, a\rangle}{\langle\psi, a\wedge b\rangle} \quad \text{and} \quad \frac{\langle\varphi, a\rangle}{\langle\varphi, b\rangle} \text{ with } b \leq a. \tag{7}$$

This gives in an obvious way a notion of provability, $\vdash$, with respect to which the following completeness theorem holds:

$$T \vdash \langle\varphi, a\rangle \text{ iff } T \models \langle\varphi, a\rangle. \tag{8}$$

Such logic may essentially be viewed as an abstract fuzzy logic $\mathsf{L}_{\mathrm{Pos}} = \langle\mathcal{F}, \langle[0,1], \leq\rangle, \mathcal{S}_{\mathrm{Pos}}, A_{\mathrm{Pos}}, \mathcal{R}_{\mathrm{Pos}}\rangle$ where $\mathcal{F}$ is the set of formulas of classical propositional logic, $\mathcal{S}_{\mathrm{pos}} = \{N_\pi \mid \pi : W \to [0,1] \text{ normalized}\}$, $A_{\mathrm{Pos}} = \{\langle\varphi_1, 1\rangle, \ldots, \langle\varphi_k, 1\rangle\}$, and $\mathcal{R}_{\mathrm{Pos}}$ consists of the rules in (7). Note that the second rule does not directly conform to Pavelka's notion of a deduction rule (b is not a function of a) but may equivalently be replaced by a collection of Pavelka's deduction rules; also note that if one is interested in the degree of provability, i.e. in searching good proofs, the rule may be disposed of. Now, $[0,1]$-weighted formulas of this abstract fuzzy logic are just the possibilistic formulas (except that in possibilistic logic one excludes $\langle\varphi, a\rangle$ with $a = 0$). Clearly, whether a theory is a set T of pairs $\langle\varphi, a\rangle$ (as in possibilistic logic) or a fuzzy set T of formulas such that $T(\varphi) = a$ (as in abstract fuzzy logic) is clearly a *façon de parler*, hence theories in possibilistic logic may be identified with theories in Pavelka's sense. Note now that the relation $T \models \langle\varphi, a\rangle$ is worked with but usually not explicitly denoted in abstract fuzzy logic and that $\mathrm{Val}(\varphi, T)$ is indeed an ordinary notion of abstract fuzzy logic:

Observation 1. $\mathrm{Val}(\varphi, T)$ *equals the degree* $\|\varphi\|_T$ *of semantic entailment of* φ *from* T *in* $\mathsf{L}_{\mathrm{Pos}}$.

Proof. The equality follows from definitions. $\mathrm{Val}(\varphi, T) \leq \|\varphi\|_T$: We need to see that if $T \models \langle\varphi, a\rangle$ then $a \leq N_\pi(\varphi)$ for any $N_\pi \in \mathrm{Mod}(T)$. But if $N_\pi \in \mathrm{Mod}(T)$, then $a \leq N_\pi(\varphi)$ directly follows from $T \models \langle\varphi, a\rangle$.

[14] For other aspects, see Gerla's [13, chapter 6]. Note that Gerla naturally arrived at possibilistic logic within his studies of abstract fuzzy logic.

[15] Lang in fact obtained an axiomatization of predicate possibilistic logic.

$\|\varphi\|_T \leq \mathrm{Val}(\varphi, T)$: We prove this by checking that $\|\varphi\|_T$ is one the as for which $T \models \langle\varphi, a\rangle$. This amounts to showing that if $N_\pi \in \mathrm{Mod}(T)$, then $\|\varphi\|_{N_\pi} \geq \|\varphi\|_T$, which immediately follows from the definition of $\|\varphi\|_T$. □

Observe that in terms of abstract fuzzy logic, $T \vdash \langle\varphi, a\rangle$ just means that there exists a [0, 1]-weighted proof of $\langle\varphi, a\rangle$ from T. Before presenting the next theorem, note that having *optimal proofs* [13] means that for each T and φ, the degree $|\varphi|_T$ of provability is actually attained by some proof, i.e. there exists a weighted proof of $\langle\varphi, |\varphi|_T\rangle$ from T. The next theorem, which in particular implies that our logic $\mathsf{L}_{\mathrm{Pos}}$ is Pavelka-style complete and has optimal proofs, closes our visit to possibilistic logic:

Observation 2. *The completeness theorem of possibilistic logic, (8), is equivalent to the claim that* $\mathsf{L}_{\mathrm{Pos}}$ *is Pavelka-style complete, i.e.* $|\varphi|_T = \|\varphi\|_T$, *and has optimal proofs.*

Proof. The ⇒-part follows from Lemma 1 and the definitions of $|\varphi|_T$. The ⇐-part is immediate by realizing that the second of the rules in (7) is available. □

The above example further demonstrates that Pavelka's abstract fuzzy logic is a powerful framework, capable of accommodating various systems of approximate reasoning. Still, as shown above, several misconceptions regarding it prevail. The primary reason, I believe, is that Pavelka's conception is not properly portrayed in the literature. Part of the purpose of the present paper is help make this portrayal somewhat clearer.

Acknowledgment

Supported by grant No. GA14-11585S of the Czech Science Foundation.

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