Representation of Concept Lattices by Bidirectional Associative Memories

Radim Bělohlávek
Institute for Research and Application of Fuzzy Modeling, University of Ostrava, CZ-701 03 Ostrava, Czech Republic and Department of Computer Science, Technical University of Ostrava, CZ-708 33 Ostrava, Czech Republic

This article presents a concept interpretation of patterns for bidirectional associative memory (BAM) and a representation of hierarchical structures of concepts (concept lattices) by BAMs. The constructive representation theorem provides a storing rule for a training set that allows a concept interpretation. Examples demonstrating the theorems are presented.

1 Introduction and Preliminaries

Two levels have to be taken into account when modeling intelligent systems: the microlevel and the macrolevel. This is because there are two corresponding levels evidenced by biological systems, which are supposed to be intelligent: the level of brain and the level of mental phenomena. The macrolevel (mental level) is supposed to be implemented in the microlevel (brain level). The inspiration from the microlevel gave rise to the paradigm of artificial neural networks: systems based on similar principles as their biological counterparts. On the other hand, there are many models inspired by the macrolevel. A challenging goal is the development of architectures that exhibit both of levels. On the macrolevel, a clear interpretation of the system is possible, while on the microlevel, an analysis up to an appropriate degree of exactness can be performed.

This article deals with a macrolevel interpretation of the bidirectional associative memory (BAM) (Kosko, 1987, 1988). The interpretation is in terms of concepts: BAM patterns are interpreted to represent concepts in the sense of Wille (1982) and Ganter and Wille (1999). Sections 2 and 3 survey BAMs and fundamentals of concept lattices, respectively. In section 4, a concept interpretation of BAM patterns is proposed and discussed, and a storing rule and representation theorem are presented. Section 5 contains illustrative examples.

2 Bidirectional Associative Memories

Associative memories represent a class of neural networks that aim at modeling the association phenomenon (Arbib, 1995; Bělohlávek, 1998). Based on

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the early models of Amari (1972) and Hopfield (1984), Kosko (1987, 1988) proposed a bidirectional associative neural network called bidirectional associative memory. BAM consists of two layers of neurons. The first and the second layers contain k and l neurons, respectively, states (signals) of which are denoted by x_i ($i = 1, \ldots, k$) and y_j ($j = 1, \ldots, l$). The states x_i and y_j take the values $-1, 1$ by bipolar encoding and $0, 1$ by binary encoding (to which we restrict ourselves). Each (ith) neuron of the first layer is connected to each (jth) neuron of the second layer by a connection with the real weight w_{ij}. A real threshold θ_i^x (θ_j^y) is assigned to the ith neuron of the first layer (jth neuron of the second layer). The dynamics is bidirectional: given a pair $\langle X, Y\rangle = \langle\langle x_1, \ldots, x_k\rangle, \langle y_1, \ldots, y_l\rangle\rangle \in \{0, 1\}^k \times \{0, 1\}^l$ of patterns of signals, the signal is fed to the second layer to obtain a new pair $\langle X, Y'\rangle$, then again to the first layer to obtain $\langle X', Y'\rangle$, and so on. The dynamics is given by the formulas

$$y'_j = \begin{cases} 1 & \text{for } \sum_{i=1}^{k} w_{ij}x_i > \theta_j^y \\ y_j & \text{for } \sum_{i=1}^{k} w_{ij}x_i = \theta_j^y \\ 0 & \text{for } \sum_{i=1}^{k} w_{ij}x_i < \theta_j^y, \end{cases} \qquad x'_i = \begin{cases} 1 & \text{for } \sum_{j=1}^{l} w_{ij}y'_j > \theta_i^x \\ x_i & \text{for } \sum_{j=1}^{l} w_{ij}y'_j = \theta_i^x \\ 0 & \text{for } \sum_{j=1}^{l} w_{ij}y'_j < \theta_i^x. \end{cases} \tag{2.1}$$

The pair of patterns $\langle X, Y\rangle$ is called a stable point if the states of neurons, which are set to $\langle X, Y\rangle$, do not change under the defined dynamics. The set of all stable points of a BAM will be denoted by $Stab(W, \Theta)$. Using appropriate energy function, Kosko (1988) proved that such a network is stable for any weights w_{ij} and any thresholds θ_i^x, θ_j^y.[1] Stability means that given any initial pattern $\langle X, Y\rangle$ of signals, the net eventually stops after a finite number of steps (feeding signal from layer to layer).

The aim of learning in the context of associative memories is to set the parameters of the net so that a prescribed training set of patterns is related in some way to the set of all stable points. Usually all training patterns have to become stable points. Kosko proposes a kind of Hebbian learning, by which the weights w_{ij} are determined from the training set $T = \{\langle X^p, Y^p\rangle \mid p \in P\}$ by

$$w_{ij} = \sum_{p\in P} \mathrm{bip}\left(x_i^p\right) \cdot \mathrm{bip}\left(y_j^p\right), \tag{2.2}$$

where bip maps 1 to 1 and 0 to -1, that is, it changes the binary encoding to a bipolar one. Thresholds are set to 0. Another rule has been proposed by Wang, Cruz, and Mulligan (1990), where a further theoretical analysis of BAM dynamics is provided.

[1] In fact, Kosko proved the stability theorem for $\theta_i^x = \theta_j^y = 0$. It is a matter of routine to extend the proof to arbitrary θ_i^x and θ_j^y.

3 Concepts and Concept Lattices

The notion of concept, central in human thinking, appears as well in the context of neural networks. Networks are seen as if extracting characteristic features of input data that are considered to represent concepts. Attributes other than the aggregation function (selection of characteristic features) are usually ignored.

In a programmatic paper Wille (1982) formulated the theory of concept lattices, which serves as the foundation of formal concept analysis (Ganter & Wille, 1999). It is based on a traditional understanding of concepts (the Port Royal school) by which a concept is determined by its extent and its intent. The extent of a concept (e.g., DOG) is the collection of all objects covered by the concept (the collection of all dogs), while the intent is the collection of all attributes (e.g., "to bark," "to be a mammal") covered by the concept. The starting point of the formalization is that of context: a triple $\langle G, M, I\rangle$, where I is a binary relation between G and M, that is, $I \subseteq G \times M$. Elements of G are interpreted as objects, elements of M as attributes, and the fact $\langle g, m\rangle \in I$ as "the object g has the attribute m." According to philosophical tradition, a (formal) concept in a given context is any pair $\langle A, B\rangle$ of extent $A \subseteq G$ and intent $B \subseteq M$ such that $A = B^{\downarrow} :=_{\text{def}} \{g \in G \mid \text{for all } m \in B \text{ it holds } \langle g, m\rangle \in I\}$ and $B = A^{\uparrow} :=_{\text{def}} \{m \in M \mid \text{for all } g \in A \text{ it holds } \langle g, m\rangle \in I\}$. In other words, $\langle A, B\rangle$ is a concept if $A = B^{\downarrow}$ and $B = A^{\uparrow}$; that is, A is the set of all objects $g \in G$ that have all the attributes m of B, and, conversely, B is the set of all attributes $m \in M$ that are shared by all the objects g of A.

The crucial relation between concepts is that of a hierarchical ordering. The hierarchy of concepts plays a fundamental role in conceptual reasoning. Denote $\mathcal{B}(G, M, I)$ the set of all concepts in the context $\langle G, M, I\rangle$,

$$\mathcal{B}(G, M, I) = \{\langle A, B\rangle \mid A = B^{\downarrow}, B = A^{\uparrow}\},$$

and for $\langle A_1, B_1\rangle, \langle A_2, B_2\rangle \in \mathcal{B}(G, M, I)$ put $\langle A_1, B_1\rangle \leq \langle A_2, B_2\rangle$ iff $A_1 \subseteq A_2$ (which is equivalent to $B_1 \supseteq B_2$). The relation $\leq$ naturally models the relation "to be a subconcept," and $\langle A_1, B_1\rangle \leq \langle A_2, B_2\rangle$ means that each object covered by $\langle A_1, B_1\rangle$ is also covered by $\langle A_2, B_2\rangle$ (as an example, consider the concepts DOG and MAMMAL). The fundamental structure of the set of all concepts given by a context is given by the following proposition (Wille, 1982), which is part of the so-called main theorem of conceptual data analysis.

Proposition. *Let $\langle G, M, I\rangle$ be a context. Then the set $\mathcal{B}(G, M, I)$ is under the relation $\leq$ introduced above a complete lattice where*

$$\bigwedge\{\langle A_j, B_j\rangle;\ j \in J\} = \left\langle \bigcap\{A_j;\ j \in J\}, \left(\bigcap\{A_j;\ j \in J\}\right)^{\uparrow} \right\rangle,$$

$$\bigvee\{\langle A_j, B_j\rangle;\ j \in J\} = \left\langle \left(\bigcap\{B_j;\ j \in J\}\right)^{\downarrow}, \bigcap\{B_j;\ j \in J\} \right\rangle.$$

Recall that a complete lattice (Birkhoff, 1967) is a partially ordered set in which each subset has the least upper bound (supremum) and the greatest lower bound (infimum). The lattice $\mathcal{B}(G, M, I)$ is called a concept lattice given by the context $\langle G, M, I\rangle$. The complete lattice structure is a very natural one for conceptual structures. Informally, it states that for each set of concepts, there is a direct generalization (supremum) as well as a direct specialization (infimum). Note that the visualizable hierarchical structure of the revealed concepts is the basic tool of conceptual data analysis. Further results can be found in Ganter and Wille (1999).

4 Representation and Storing of Concept Lattices

Our aim now is to provide a conceptual interpretation of BAM and to show that BAMs can represent lattices of concepts. To this end, we accept the following convention. For a set $Z = \{z_1, \ldots, z_n\}$ and a subset $A \subseteq Z$, we denote by $s_Z(A) = \langle a_1, \ldots, a_n\rangle \in \{0, 1\}^n$ the characteristic vector of A, that is, $a_i = 1$ if $z_i \in A$ and $a_i = 0$ if $z_i \notin A$.

Let us have a context $\langle G, M, I\rangle$ with both G and M finite, $G = \{g_1, \ldots, g_k\}$, $M = \{m_1, \ldots, m_l\}$. Using the grandmother cell idea (Arbib, 1995) we can consider a BAM with k neurons in the first layer and l neurons in the second layer, with the following interpretation: The states of both of the layers represent subsets of G and M, respectively. The set $A \subseteq G$ is represented by the vector $s_G(A) \in \{0, 1\}^k$ of the states of the first layer, and similarly $B \subseteq M$ is represented by $s_M(B) \in \{0, 1\}^l$. The ith neuron of the first layer therefore represents the object g_i, and the jth neuron of the second layer represents the property m_j. The pairs of subsets of G and M are thus in a one-to-one correspondence with the pairs of states of the first and the second layers.

Example 1. In general, the concept interpretation of the patterns of states is not possible. We easily find a BAM stable points, but they cannot be interpreted as concepts. As an example, consider $k = 1, m = 2, w_{11} = 1$, and $w_{12} = -2$, all thresholds set to 0. Then $Stab(W, \Theta) = \{\langle 0, \langle 0, 0\rangle\rangle, \langle 0, \langle 0, 1\rangle\rangle, \langle 0, \langle 1, 1\rangle\rangle, \langle 1, \langle 1, 0\rangle\rangle\}$. For $\langle s_G(A_1), s_M(B_1)\rangle = \langle 0, \langle 0, 0\rangle\rangle$, $\langle s_G(A_2), s_M(B_2)\rangle = \langle 1, \langle 1, 0\rangle\rangle$, we have $A_1 \subset A_2$ but $B_1 \not\supseteq B_2$ which contradicts the rule that the more common objects there are, the fewer the common properties, which is valid for formal concepts.

On the other hand, taking, for example, $k = l = 2$, $w_{11} = w_{12} = 1$, $w_{12} = w_{21} = -3$, all thresholds set to $-\frac{1}{2}$. Then $Stab(W, \Theta) = \{\langle\langle 0, 0\rangle, \langle 1, 1\rangle\rangle, \langle\langle 0, 1\rangle, \langle 0, 1\rangle\rangle, \langle\langle 1, 0\rangle, \langle 1, 0\rangle\rangle, \langle\langle 1, 1\rangle, \langle 0, 0\rangle\rangle\}$. As can be easily verified, $Stab(W, \Theta)$ corresponds to $\mathcal{B}(G, M, I)$ for $I = \{\langle g_1, m_1\rangle, \langle g_2, m_2\rangle\}$ by $Stab(W, \Theta) = \{\langle s_G(A), s_M(B)\rangle \mid \langle A, B\rangle \in \mathcal{B}(G, M, I)\}$.

The crucial question therefore is: Is there for each concept lattice $\mathcal{B}(G, M, I)$ a BAM such that the set of all concepts of $\mathcal{B}(G, M, I)$ is (modulo the above correspondence) precisely the set of all stable points of this BAM? A positive answer is given by the following theorem.

Theorem 1. *Let $\mathcal{B}(G, M, I)$ be a concept lattice given by the context $\langle G, M, I\rangle$ with G and M finite. Then there is a BAM given by the weights W and thresholds Θ such that $Stab(W, \Theta) = \{\langle s_G(A), s_M(B)\rangle \mid \langle A, B\rangle \in \mathcal{B}(G, M, I)\}$; that is, there is a one-to-one correspondence between stable points and formal concepts.*

Proof. Let $G = \{g_1, \ldots, g_k\}, M = \{m_1, \ldots, m_l\}$. Define a BAM by the matrix W given by

$$w_{ij} = \begin{cases} 1 & \text{if } \langle g_i, m_j\rangle \in I \\ -q & \text{if } \langle g_i, m_j\rangle \notin I \end{cases} \tag{4.1}$$

for $i = 1, \ldots, k, j = 1, \ldots, l$ where $q = \max\{k, l\} + 1$. Set all the thresholds to $-\frac{1}{2}$. Denote by $x(t)$ and $y(t)$, $t = 0, 1, 2, \ldots,$ the vectors of the states of the first and the second layers, respectively, at time t.

Let $\langle A, B\rangle \in \mathcal{B}(G, M, I)$. We have to show that $\langle s_G(A), s_M(B)\rangle$ is a stable point of the BAM. To this end, initialize the network with $\langle s_G(A), s_M(B)\rangle$, that is, $x(0) = \langle x_1(0), \ldots, x_k(0)\rangle = s_G(A), y(0) = \langle y_1(0), \ldots, y_l(0)\rangle = s_M(B)$. We show $\langle x(1), y(1)\rangle = \langle x(0), y(0)\rangle$; feeding the signal forward, the states do not change. Clearly $x(1) = x(0)$ (if the signal is fed forward, the first layer does not change its state). Consider now any y_j. We distinguish two cases. First, let $y_j(0) = 1$. Since $\langle A, B\rangle$ is a concept, we have $A^{\uparrow} = B$, and so $\langle g_i, m_j\rangle \in I$ (i.e., $w_{ij} = 1$ by equation 4.1) for each i such that $g_i \in A$ (i.e., such that $x_i(0) = 1$). We therefore have

$$\sum_{i=1}^{k} w_{ij}x_i(0) = |A| \geq -\frac{1}{2} = \theta_j^y.$$

By the activation dynamics (see equation 2.1) of BAM, we have $y_j(1) = 1$. For $y_j(0) = 1$, the state therefore does not change. Second, let $y_j(0) = 0$. Since $\langle A, B\rangle$ is a concept, there is some i such that $g_i \in A$ (i.e., $x_i(0) = 1$) but $\langle g_i, m_j\rangle \notin I$ (i.e., $w_{ij} = -q$). Denote by K the set of all such i. Denote furthermore by K^* the set of i such that $g_i \in A$ (i.e., $x_i(0) = 1$) and $\langle g_i, m_j\rangle \in I$ (i.e., $w_{ij} = 1$). We have

$$\sum_{i=1}^{k} w_{ij}x_i(0) = \sum_{i\in K} w_{ij}x_i(0) + \sum_{i\in K^*} w_{ij}x_i(0) = -|K|q + |K^*| < -\frac{1}{2},$$

since $q > k \geq |K^*|$, $|K| \geq 1$, and $-|K|q + |K^*|$ is an integer. By the BAM dynamics, we have $y_j(1) = 0$; the state does not change. We have proved $y(1) = y(0)$. We should now show that $x(2) = x(1)$ by the backward phase.

The proof is completely symmetric and therefore will be omitted. We have thus proved that $\langle s_G(A), s_M(B)\rangle$ is stable.

Conversely, let $\langle s_G(A), s_M(B)\rangle = \langle x, y\rangle \in Stab(W, \Theta)$. We have to show that $\langle A, B\rangle$ is a concept of $\mathcal{B}(G, M, I)$, that is, $A^{\uparrow} = B$ and $B^{\downarrow} = A$. Again, due to symmetry, we show only $A^{\uparrow} = B$. We reason as follows: $m_j \in A^{\uparrow}$ iff for each i such that $g_i \in A$ (i.e., $x_i = 1$) we have $\langle g_i, m_j\rangle \in I$ (i.e., $w_{ij} = 1$). The last assertion holds iff $\sum_{i=1}^{k} w_{ij}x_i > -\frac{1}{2}$. (Indeed, if for each i such that $x_i = 1$ it holds that $w_{ij} = 1$, then $\sum_{i=1}^{k} w_{ij}x_i \geq 0 > -\frac{1}{2}$.)

The direction $\Rightarrow$ is clear. Conversely, let $\sum_{i=1}^{k} w_{ij}x_i > -\frac{1}{2}$. If there would be some i such that $g_i \in A$ ($x_i = 1$) and $\langle g_i, m_j\rangle \notin I$ ($w_{ij} = -q$), then from $q > k$ we conclude $\sum_{i=1}^{k} w_{ij}x_i < -\frac{1}{2}$, a contradiction. By the BAM dynamics equation 2.1, $\sum_{i=1}^{k} w_{ij}x_i > -\frac{1}{2}$ iff $y_j = 1$. To sum up, $m_j \in A^{\uparrow}$ iff $y_j = 1$, hence $A^{\uparrow} = B$. The theorem is proved.

Corollary 1. *For each finite lattice $\mathcal{L} = \langle L, \preceq\rangle$, there is a BAM given by the weights W and thresholds Θ such that under the relation $\leq$ defined on $Stab(W, \Theta)$ by*

$$\langle x_1, y_1\rangle \leq \langle x_2, y_2\rangle \quad \text{iff} \quad x_{1i} \leq x_{2i}\ (\forall i)\ \big(\text{iff } y_{1j} \geq y_{2j}\ (\forall j)\big) \tag{4.2}$$

$\langle Stab(W, \Theta), \leq\rangle$ and $\langle L, \preceq\rangle$ are isomorphic lattices.

Proof. By the first theorem of Wille (1982), the lattice $\mathcal{L}$ is isomorphic to the concept lattice $\mathcal{B}(L, L, \preceq)$ (i.e., one puts $G = M = L$ and $I = \preceq$). Denote the isomorphism by φ. By theorem 1, there is a BAM given by W, Θ and a one-to-one correspondence ψ between $\mathcal{B}(L, L, \preceq)$ and $Stab(W, \Theta)$. It is obvious that if one introduces the relation $\leq$ on $Stab(W, \Theta)$ by equation 4.2, ψ becomes an isomorphism between $\langle \mathcal{B}(L, L, \preceq), \leq\rangle$ and $\langle Stab(W, \Theta), \leq\rangle$. Therefore, the composite mapping $\varphi \circ \psi$ is an isomorphism between $\mathcal{L} = \langle L, \preceq\rangle$ and $\langle Stab(W, \Theta), \leq\rangle$.

The proof of theorem 1 gives also the rule (equation 4.1) for storing the set of all patterns (concepts) $\langle A, B\rangle$ of $\mathcal{B}(G, M, I)$ from the relation I. However, one might be concerned with a situation where the training information is not in the form of a binary relation but in the form of a training set

$$T = \{\langle A^p, B^p\rangle \mid A \subseteq G,\ B \subseteq M,\ p \in P\}$$

of patterns. In this case, the fundamental question is whether T can be interpreted as a (consistent) structure of concepts. Call T conceptually consistent if there is a concept lattice $\mathcal{B}(G, M, I)$ such that $T \subseteq \mathcal{B}(G, M, I)$, that is, each $\langle A, B\rangle \in T$ is a concept of a fixed concept lattice. In the following we concentrate on the problem of finding necessary and sufficient conditions for a training set T to be conceptually consistent and on the problem of storing a conceptually consistent training set.

Lemma 1. *Let $\mathcal{B}(G, M, I)$ be a concept lattice and put $T = \mathcal{B}(G, M, I)$. Define the relation $I_T \subseteq G \times M$ by:*

$$\langle g, m\rangle \in I_T \quad \text{iff there exists} \quad \langle A, B\rangle \in T \text{ such that } g \in A, m \in B. \tag{4.3}$$

Then it holds $I = I_T$.

Proof. If $\langle g, m\rangle \in I_T$, then by equation 4.3, there is some $\langle A, B\rangle \in T$ such that $g \in A, m \in B$. Since $T = \mathcal{B}(G, M, I)$, $\langle A, B\rangle$ is a formal concept, therefore $A^{\uparrow} = B$. By definition of $A^{\uparrow}$, we conclude that $\langle g, m\rangle \in I$, that is, $I_T \subseteq I$. Conversely, if $\langle g, m\rangle \in I$, then for $\langle A, B\rangle = \langle \{g\}^{\uparrow\downarrow}, \{g\}^{\uparrow}\rangle \in \mathcal{B}(G, M, I)$, we have $g \in A, m \in B$, that is, $\langle g, m\rangle \in I_T$, which proves $I \subseteq I_T$.

Lemma 2. *For a conceptually consistent T, let I_T be defined by equation 4.3. Then it holds that $T \subseteq \mathcal{B}(G, M, I_T)$.*

Proof. If T is conceptually consistent, then $T \subseteq \mathcal{B}(G, M, I)$ for some $I \subseteq G \times M$. By the definition of equation 4.3 of I_T and lemma 1, $I_T \subseteq I$. Denote the operators corresponding to I and I_T by $^{\uparrow}$, $^{\downarrow}$ and $^{\uparrow_T}$, $^{\downarrow_T}$, respectively. Take $\langle A, B\rangle \in T$. Since $I_T \subseteq I$, we have $B = A^{\uparrow} = \{m \in M \mid \forall g \in A\colon \langle g, m\rangle \in I\} \supseteq \{m \in M \mid \forall g \in A\colon \langle g, m\rangle \in I_T\} = A^{\uparrow_T}$. On the other hand, since for every $g \in A, m \in B$ we have $\langle g, m\rangle \in I_T$ (by equation 4.3), it holds $A^{\uparrow_T} \supseteq B$, that is, $A^{\uparrow_T} = B$). One would similarly obtain $B^{\downarrow_T} = A$. To sum up, $A^{\uparrow_T} = B$ and $B^{\downarrow_T} = A$, that is, $\langle A, B\rangle \in \mathcal{B}(G, M, I_T)$. We have proved $T \subseteq \mathcal{B}(G, M, I_T)$.

We have the following criterion for a training set to be conceptually consistent.

Theorem 2. *A training set $T = \{\langle A^p, B^p\rangle \mid p \in P\}$ is conceptually consistent iff for each $p \in P$ it holds*

$$A^p = \{g \in G \mid \forall m \in B^p\, \exists p' \in P\colon g \in A^{p'}, m \in B^{p'}\}$$
$$B^p = \{m \in M \mid \forall g \in A^p\, \exists p' \in P\colon g \in A^{p'}, m \in B^{p'}\}.$$

Proof. If T is conceptually consistent, then, by lemma 2, $T \subseteq \mathcal{B}(G, M, I_T)$. Therefore, each $\langle A^p, B^p\rangle \in T$ is a formal concept, and thus $A^{p\uparrow} = B^p$. By definition of $A^{p\uparrow}$ and equation 4.3, this means that $B^p = \{m \in M \mid \forall g \in A^p\colon \langle g, m\rangle \in I_T\} = \{m \in M \mid \forall g \in A^p\, \exists p' \in P\colon g \in A^{p'}, m \in B^{p'}\}$, which is the second equality to be proved. The first equality can be obtained symmetrically. Conversely, if for each $p \in P$ the above equalities hold, then

$T \subseteq \mathcal{B}(G, M, I_T)$, since the equalities just express the facts $A^p = B^{p\downarrow}$ and $B^p = A^{p\uparrow}$, respectively.

By the previous results, we have for a training set $T = \{\langle A^p, B^p\rangle \mid A \subseteq G, B \subseteq M, p \in P\}$ with $|G| = k$, $|M| = l$ the following storing algorithm: For $i = 1, \ldots, k$, $j = 1, \ldots, l$, set the weights by

$$w_{ij} = \begin{cases} 1 & \text{if } \exists p \in P\colon\ g \in A^p,\ m \in B^p \\ -(\max\{k, l\} + 1) & \text{otherwise} \end{cases}$$

and the thresholds by

$$\theta_i^x = -\frac{1}{2}, \qquad \theta_j^y = -\frac{1}{2}.$$

Call T *storable* (by our algorithm) if for the weights and the thresholds set according the algorithm, it holds $s(T) \subseteq Stab(W, \Theta)$ where $s(T) = \{\langle s_G(A), s_M(B)\rangle \mid \langle A, B\rangle \in T\}$. The scopes and limits of the algorithm are described by the following assertion.

Corollary 2. *A training set T is storable iff it is conceptually consistent.*

Proof. If T is conceptually consistent, then T is storable by lemma 2. Conversely, if T is storable, then, by definition, $s(T) \subseteq Stab(W, \Theta)$. The assertion follows from the fact that if W and Θ are set from T by our algorithm, then $Stab(W, \Theta) = s(\mathcal{B}(G, M, I_T))$ by theorem 1.

5 Examples

Example 2. Let a context be given by the set G of nine planets (Mercury, ..., Pluto), the set M of seven attributes ("size small," ..., "does not have a moon"), and the relation I between them depicted in Table 1 (see Wille, 1982). The BAM determined from this context by theorem 1 consists of nine and seven neurons in the first and the second layer, respectively. The set of all stable points (i.e., concepts) is depicted in Table 2. The approximate linguistic description of the stable points is as follows: 1, the empty concept; 2, "small planet without moon near to sun"; 3, "small planet with moon(s) near to sun"; 4, "small planet with moon(s) far from sun"; 5, "large planet with moon(s) far from sun"; 6, "medium planet with moon(s) far from sun"; 7, "small planet near to sun"; 8, "small planet with moon(s)"; 9, "planet far from sun"; 10, "small planet"; 11, "planet with moon"; 12, "planet." The conceptual hierarchy of the stable points (concept lattice) is depicted in Figure 1.

Table 1: Planets and Their Attributes.

	Size			Distance from Sun		Has Moon	
	Small (ss)	Medium (sm)	Large (sl)	Near (dn)	Far (df)	Yes (my)	No (mn)
Mercury (Me)	×			×			×
Venus (V)	×			×			×
Earth (E)	×			×		×	
Mars (Ma)	×			×		×	
Jupiter (J)			×		×	×	
Saturn (S)			×		×	×	
Uranus (U)		×			×	×	
Neptune (N)		×			×	×	
Pluto (P)	×				×	×	

Table 2: Stable Points of the Planet Example.

Number	Extent									Intent						
	Me	V	E	Ma	J	S	U	N	P	ss	sm	sl	dn	df	my	mn
1	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1
2	1	1	0	0	0	0	0	0	0	1	0	0	1	0	0	1
3	0	0	1	1	0	0	0	0	0	1	0	0	1	0	1	0
4	0	0	0	0	0	0	0	0	1	1	0	0	0	1	1	0
5	0	0	0	0	1	1	0	0	0	0	0	1	0	1	1	0
6	0	0	0	0	0	0	1	1	0	0	1	0	0	1	1	0
7	1	1	1	1	0	0	0	0	0	1	0	0	1	0	0	0
8	0	0	1	1	0	0	0	0	1	1	0	0	0	0	1	0
9	0	0	0	0	1	1	1	1	1	0	0	0	0	1	1	0
10	1	1	1	1	0	0	0	0	1	1	0	0	0	0	0	0
11	0	0	1	1	1	1	1	1	1	0	0	0	0	0	1	0
12	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0

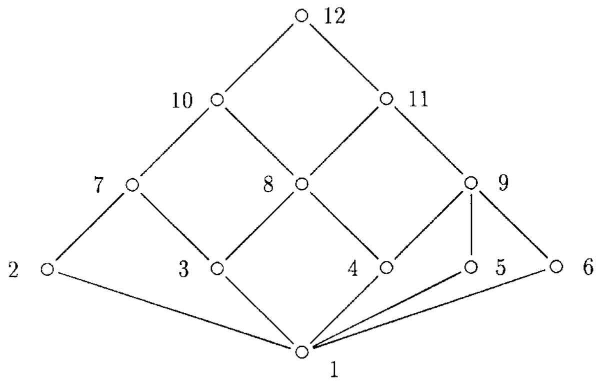

Figure 1: Hierarchical structure (lattice) of stable points of the planet example.

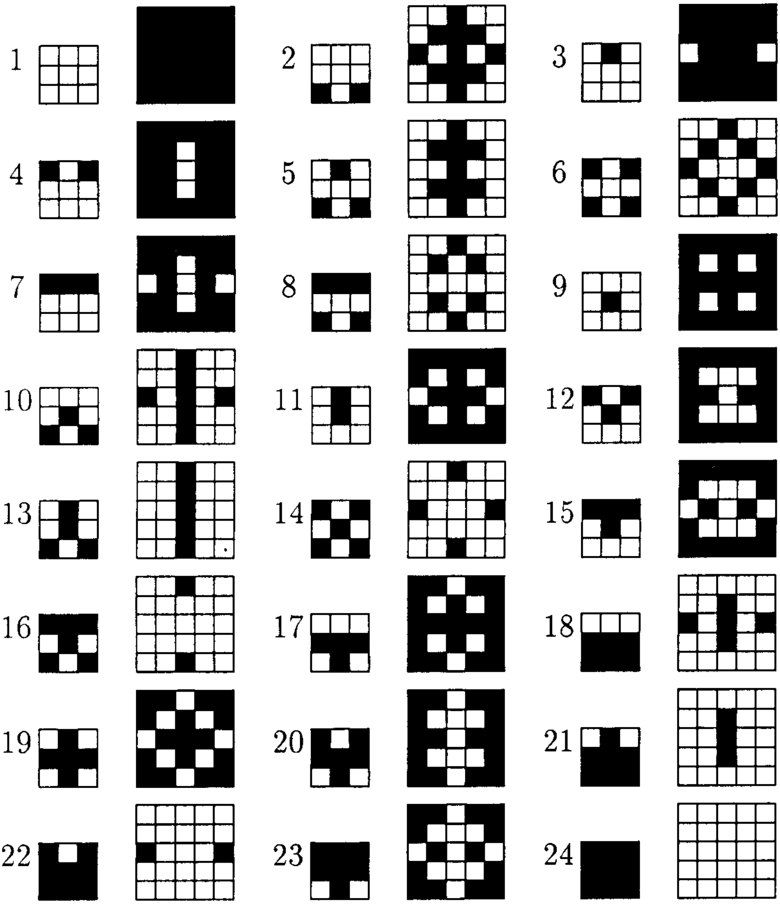

Figure 2: Stable points.

Example 3. Consider a BAM with 9 neurons in the first layer and 25 neurons in the second layer. We will represent each state $\langle X, Y\rangle$ of the net by a pair of bitmap pictures: a 3×3 bitmap for X and 5×5 bitmap for Y. If the neuron state is 0 (1), then the corresponding pixel is white (black). Consider the training set T consisting of six pairs $\langle A, B\rangle$ such that $\langle s_G(A), s_M(B)\rangle$ are represented by the pairs 5, 6, 9, 19, 22, and 23 of Figure 2. By theorem 2, it is easy to verify that T is conceptually consistent. The storing algorithm yields a BAM with 24 stable points depicted in Figure 2. The conceptual hierarchy of the stable points is visualized in Figure 3. The patterns of T have been stored and completed into a complete conceptual structure. Note that concepts 6 and 19 of the training set are complementary. The stored structure of

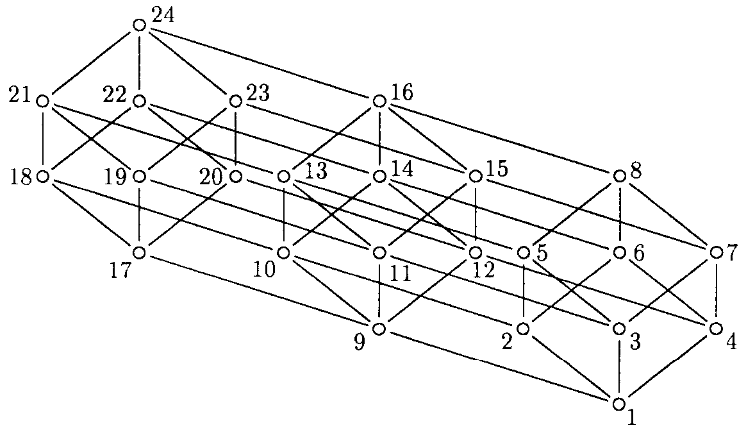

Figure 3: Hierarchical structure (lattice) of stable points.

concepts contains additional pairs of complementary concepts ("opposite concepts," in conceptual terms), for example, 5 and 20 (5 is in T) and 7 and 18 (none of them is in T). Note that the pairs of complementary concepts, viewed from the lattice point of view, form complementary elements in the lattice of concepts.

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