On the regularity of MV-algebras and Wajsberg hoops

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An algebra $\mathbf{A}$ is called *congruence regular* iff each congruence of $\mathbf{A}$ is determined by each of its classes, i.e. iff $[a]_\theta = [a]_\phi$ implies $\theta = \phi$ for every $\theta, \phi \in \operatorname{Con} \mathbf{A}$ and each $a \in A$. A variety of algebras is congruence regular iff each of its members has this property. Congruence regular varieties have been characterized in [7, 8, 10]: A variety $\mathcal{V}$ is congruence regular iff there are ternary terms $t_1, \ldots, t_n$ (referred to as regularity terms) such that $[t_1(x,y,z) = z, \ldots, t_n(x,y,z) = z]$ iff $x = y$. Examples of regular varieties are quasigroups, groups, rings, Boolean algebras. All these algebras have a single regularity term, i.e. one may put $n = 1$ in the above characterization ($y/(z\backslash x)$ for quasigroups, $x \cdot y^{-1} \cdot z$ for groups etc.). The aim of this note is to show that MV-algebras (and more generally, Wajsberg hoops) are regular but don't have a single regularity term.

MV-algebras have been introduced as the algebraic counterpart of Łukasiewicz logic [5]. An MV-algebra is an algebra $\mathbf{A} = \langle A, \oplus, \neg, 0 \rangle$ with an associative, commutative binary operation $\oplus$ with a neutral element 0, and a unary operation $\neg$ satisfying $\neg\neg x = x$, $x \oplus \neg 0 = \neg 0$, $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$. Putting $x \odot y = \neg(\neg x \oplus \neg y)$ we get a dual operation (associative, commutative, with the neutral element $1 = \neg 0$). A lattice order $\leq$ is induced in each MV-algebra by $x \leq y$ iff $\neg x \oplus y = 1$ (suprema and infima are expressible by MV-algebra operations: $x \wedge y = x \odot (\neg x \oplus y)$ and $x \vee y = (x \odot \neg y) \oplus y$).

A hoop is a partially ordered commutative (dually) residuated (dually) integral monoid $\mathbf{A} = \langle A, \oplus, 0, \leq \rangle$ (i.e. $\langle A, \oplus, 0 \rangle$ is a monoid, $\langle A, \leq \rangle$ is a poset with the least element 0, $\oplus$ is isotone w.r.t. $\leq$, and for any $a, b \in A$ there exists the least element c (denoted by $a \dot{-} b$) satisfying $a \leq b \oplus c$) such that for every $a, b \in A$ we have $a \leq b$ iff $b = a \oplus c$ for some $c \in A$ (see e.g. [3]). The class of all hoops as algebras with operations $\oplus, \dot{-}, 0$ forms a variety [3, p. 295]. Any MV-algebra is a hoop where $x \dot{-} y = x \odot \neg y$. An arbitrary hoop can be embedded into a $\langle \oplus, \dot{-}, 0 \rangle$-reduct of an MV-algebra iff it satisfies $(x \dot{-} y) \dot{-} y = (y \dot{-} x) \dot{-} x$ (combine [3, Proposition 4. 1] and [2, Proposition 1. 14]).

Theorem *The variety of all MV-algebras is regular, however, it does not have a single regularity term.*

Proof. Putting $t_1(x,y,z) = (z \oplus (x \odot \neg y)) \vee (z \oplus (\neg x \odot y))$, $t_2(x,y,z) = (z \odot (\neg x \oplus y)) \wedge (z \odot (x \oplus \neg y))$ we obtain regularity terms. Indeed, one easily verifies that $t_1(x,x,z) = (z \oplus 0) \vee (z \oplus 0) = z$ and $t_2(x,x,z) = (z \odot \neg 0) \wedge (z \odot \neg 0) = z$. On the other hand, by Chang's subdirect representation theorem [6], each MV-algebra is a subdirect product of linearly ordered MV-algebras. Therefore, to see that $t_1(x,y,z) = t_2(x,y,z) = z$ implies $x = y$, one may assume that $\mathbf{A}$ is linearly ordered. If $z = 1$ then from $t_2(x,y,1) = 1$ we infer $\neg x \oplus y = 1$ and $x \oplus \neg y = 1$, i.e. $x \le y$ and $y \le x$, thus $x = y$. If $z < 1$, then, by $t_1(x,y,z) = z$, $z \oplus (x \odot \neg y) = z < 1$ and $z \oplus (\neg x \odot y) = z < 1$. From the first inequality we get $\neg z \not\le (x \odot \neg y)$ and thus $(x \odot \neg y) < \neg z$ by linearity. Now, $(x \odot \neg y) = \neg z \wedge (x \odot \neg y) = \neg z \odot (z \oplus (x \odot \neg y)) = \neg z \odot z = 0$, therefore $\neg x \oplus y = \neg(x \odot \neg y) = 1$, i.e. $x \le y$. Similarly one obtains $y \le x$ which gives $x = y$. We have proved that the variety of all MV-algebras is regular (this fact was proved (not by finding regularity terms) as a byproduct in an unpublished paper by L. P. Belluce [1]).

If there would be a single regularity term $t(x,y,z)$ then the term $q(x,y) = t(1,x,y)$ would satisfy $q(x,y) = y$ iff $x = 1$ (a single local regularity term, see [4]). We show that such a term does not exist. Assume the contrary. Take the prototypic MV-algebra $\mathbf{A}$ with $A = [0,1]$ (real numbers between 0 and 1), $x \oplus y = \min(1, x+y)$, $\neg x = 1 - x$. Consider the cube $[0,1]^3$, the term function $q^{\mathbf{A}}$ induced by $q(x,y)$, and the function $r = \{\langle a,b,b\rangle \mid a,b \in [0,1]\}$ splitting the cube (square-cut of $[0,1]^3$). Due to $q(1,y) = y$, $q^{\mathbf{A}}$ and r intersect in the line joining the vertices $\langle 1,0,0\rangle$ and $\langle 1,1,1\rangle$. Since $q(x,y) \neq y$ for $x \neq 1$, $q^{\mathbf{A}}$ and r intersect in no other point of r. It is immediate and well-known that $q^{\mathbf{A}}$ is a continuous function. If there would be some $\langle a_1,b_1\rangle, \langle a_2,b_2\rangle \in (0,1] \times [0,1]$ such that $q^{\mathbf{A}}(a_1,b_1) < r(a_1,b_1)$ and $q^{\mathbf{A}}(a_2,b_2) > r(a_2,b_2)$ or $q^{\mathbf{A}}(a_1,b_1) > r(a_1,b_1)$ and $q^{\mathbf{A}}(a_2,b_2) < r(a_2,b_2)$ then there would be another point of intersection of $q^{\mathbf{A}}$ and r, by elementary calculus. Therefore, $q^{\mathbf{A}}(a,b)$ for $a < 1$ have to lie all strictly below or all strictly above the square-cut r. An easy inspection shows that this is impossible. □

Remark. (1) As in the case of quasigroups, groups etc., the variety of all MV-algebras is also congruence permutable, i.e. $\theta \circ \phi = \phi \circ \theta$ holds for every $\theta, \phi \in \operatorname{Con} \mathbf{A}$ and each MV-algebra $\mathbf{A}$. Indeed, the term $p(x,y,z) = (x \odot (\neg y \oplus z)) \vee ((x \oplus \neg y) \odot z)$ satisfies $p(x,y,y) = x$ and $p(x,x,y) = y$, i.e. $p(x,y,z)$ is a Mal'cev permutability term [9]. Moreover, due to the lattice structure of MV-algebras, the congruence lattice of each MV-algebra is distributive. Clearly, the non-existence of a single regularity term implies that there are no MV-algebra terms which would make the MV-algebra into a quasigroup, group, Boolean algebra etc. (in general, an algebra which has a single regularity term).

(2) A closer look at the regularity terms used in the proof reveals that they can be expressed using only hoop operations $\oplus$ and $\dot{-}$, namely, $t_1(x,y,z) = (z \oplus (x \dot{-} y)) \vee (z \oplus (y \dot{-} x))$ and $t_2(x,y,z) = (z \dot{-} (x \dot{-} y)) \wedge (z \dot{-} (y \dot{-} x))$ (note that $x \wedge y = x \dot{-} (x \dot{-} y)$ and $x \vee y = (x \dot{-} y) \oplus y$). Due to the above mentioned embedding property we therefore have a stronger result: the variety of all Wajsberg hoops is congruence regular (and does not have a single regularity term).

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