Fundamenta Informaticae XXI (2001) 1001–1010
IOS Press

Reduction and a Simple Proof of Characterization of Fuzzy Concept Lattices

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Abstract. Presented is a reduction of fuzzy Galois connections and fuzzy concept lattices to (crisp) Galois connections and concept lattices: each fuzzy concept lattice can be viewed as a concept lattice (in a natural way). As a result, a simple proof of the characterization theorem for fuzzy concept lattices is obtained. The reduction enables us to apply the results worked out for concept lattices to fuzzy concept lattices.

Keywords: fuzzy concept lattice, fuzzy Galois connection

1. Introduction

The idea of extraction of information from a given data (recently referred to as data mining) has a long tradition. Coming to the question of the form of the extracted information one certainly feels as appealing the idea of methods producing information in the form of human reasoning-like structures. An interesting method of this kind is the formal concept analysis (called also the theory of concept lattices) being developed since the early 1980's by a group led by R. Wille at TU Darmstadt (see [17] for the first paper and [9] for mathematical foundations). The input data (so called formal context) is a binary relation between a set of objects and a set of attributes. The main goal is to reveal the hierarchical structure of formal concepts (in the

*Supported by grant GAČR 201/99/P060. Paper written during author's visit at the Center for Intelligent Systems, SUNY-Binghamton; support by the Center and its director, Professor G.J.Klir is gratefully acknowledged.

sense of Port-Royal logic) hidden in the input data and to investigate the dependencies among attributes. From the point of view of mathematics, although the basic construction yielding the structure of concepts is not new (it is the construction of Birkhoff's lattice of closed sets of a given polarity [7]), several interesting lattice-theoretical results have been obtained [9].

It is appealing from the point of view of fuzzy set theory [18] that the input relation between objects and attributes be fuzzy rather than sharp (two-valued). A first attempt to generalize the basic notions and results has been undertaken in [8]. Independently, a more general approach (in that the structure of truth values forms a complete residuated lattice) has been proposed and further pursued in [2, 3, 4, 5] (see also the survey in [6]).

The basic result of the theory of concept lattices, the so called Main theorem of concept lattices [17], characterizes the structure of formal concepts in a given formal context. This result has been generalized for the fuzzy case in [4]. The main aim of this paper is to present a reduction of fuzzy concept lattices to (two-valued) concept lattices. More precisely, we show that each fuzzy concept lattice can be viewed as a (two-valued) concept lattice. The idea results into a simple proof of the generalized version [4] of the Main theorem. In fact, we prove a bit more general statements about fuzzy Galois connections. As a consequence, several results of the theory of concept lattices can be almost directly applied to fuzzy concept lattices. However, this does not mean that all questions about fuzzy concept lattices are reducible to questions about concept lattices. Namely, in the fuzzy case there are several relevant phenomena (e.g. similarity, logical precision etc. [5, 6]) which are degenerate, and therefore hidden, in the case of (two-valued) concept lattices.

2. Fuzzy concept lattices

A fuzzy set [18] is a mapping from a universal set into an appropriate structure of truth values. We use complete residuated lattices as structures of truth values.

Definition 2.1. A *complete residuated lattice* is an algebra $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ such that

(1) $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice with the least element 0 and the greatest element 1,
(2) $\langle L, \otimes, 1\rangle$ is a commutative monoid, i.e. $\otimes$ is commutative, associative, and $x \otimes 1 = x$ holds holds for each $x \in L$, and
(3) $\otimes, \rightarrow$ form an adjoint pair, i.e.

$$x \otimes y \leq z \text{ iff } x \leq y \rightarrow z \tag{1}$$

holds for all $x, y, z \in L$.

For properties of complete residuated lattices we refer to [11, 13], for their role in fuzzy logic (in narrow sense) we refer to [12, 14, 15].

The most studied and applied set of truth values is the real interval $[0, 1]$ with $a \wedge b = \min(a, b)$, $a \vee b = \max(a, b)$, and with three important pairs of adjoint operations: the Łukasiewicz one ($a \otimes b = \max(a+b-1, 0)$, $a \rightarrow b = \min(1-a+b, 1)$), Gödel one ($a \otimes b = \min(a, b)$, $a \rightarrow b = 1$

if $a \leq b$ and $= b$ else), and product one ($a \otimes b = a \cdot b$, $a \to b = 1$ if $a \leq b$ and $= b/a$ else). For the role of these "building stones" in fuzzy logic see [12]. Another important set of truth values is the set $\{a_0 = 0, a_1, \ldots, a_n = 1\}$ ($a_0 < \cdots < a_n$) with $\otimes$ given by $a_k \otimes a_l = a_{\max(k+l-n,0)}$ and the corresponding $\to$ given by $a_k \to a_l = a_{\min(n-k+l,n)}$. A special case of the latter algebras is the Boolean algebra $\mathbf{2}$ of classical logic with the support $2 = \{0, 1\}$. Moreover, each left-continuous t-norm $\otimes$ (i.e. a mononotone, commutative, associative operation on $[0, 1]$ with 1 as the unit) makes $[0, 1]$ a complete residuated lattice by putting $a \to b = \bigvee\{c \mid a \otimes c \leq b\}$. Note that each of the preceding residuated lattices is complete.

A nonempty subset $K \subseteq L$ is called a *$\leq$-filter* if for every $a, b \in L$ such that $a \leq b$ it holds that $b \in K$ whenever $a \in K$. Unless otherwise stated, in what follows we denote by $\mathbf{L}$ a complete residuated lattice and by K a $\leq$-filter in $\mathbf{L}$.

An **L**-*set* (fuzzy set) [18, 10] A in a universe set X is any map $A : X \to L$. The element $A(x) \in L$ is interpreted as the truth value of the fact "x belongs to A". The concept of **L**-relation is defined obviously. By L^X we denote the set of all **L**-sets in X. Operations on L extend pointwise to L^X, e.g. $(A \vee B)(x) = A(x) \vee B(x)$ for $A, B \in L^X$. Following common usage, we write $A \cup B$ instead of $A \vee B$, etc. Given $A, B \in L^X$, the *subsethood degree* [10] $\mathrm{S}(A, B)$ of A in B is defined by $\mathrm{S}(A, B) = \bigwedge_{x \in X} A(x) \to B(x)$. We write $A \subseteq B$ if $\mathrm{S}(A, B) = 1$. Clearly, **2**-sets are the characteristic functions of (classical) sets. In the following we identify **2**-sets with sets. By $\{ {}^a/x\}$ (where $a \in L$, $x \in X$) we denote a so-called singleton, i.e. an **L**-set A in X such that $A(x) = a$ and $A(y) = 0$ for $y \neq x$.

A *(formal)* **L**-*context* (fuzzy context) is a tripple $\langle X, Y, I \rangle$ where I is a binary **L**-relation between the sets X and Y, i.e. $I \in L^{X \times Y}$. X, Y, and I are interpreted as the set of objects, the set of attributes, and the relation "to have", i.e. $I(x, y)$ is the truth degree of the fact that the object x has the attribute y. By Port-Royal logic [1], a concept is determined by its extent, i.e. a collection of all objects covered by the concept, and by its intent, i.e. a collection of all attributes covered by the concept. The extent and intent of a concept have to satisfy the following conditions: (a) the intent is the collection of all attributes shared by all objects of the extent, and (b) the extent is the collection of all objects having all the attributes of the intent. As an example, the extent of the concept DOG is the collection of all dogs, while its intent is the collection of all attributes of dogs such as "to be a mammal", "to bark" etc. Let $A \in L^X$ be a fuzzy set of objects. A straightforward consideration shows (see e.g. [3, 6]) that the fuzzy set of al attributes of Y shared by all objects of A is the fuzzy set $A^{\uparrow_I} \in L^Y$ given by

$$A^{\uparrow_I}(y) = \bigwedge_{x \in X} (A(x) \to I(x, y)). \tag{2}$$

Similarly, given a fuzzy set $B \in L^Y$ of attributes, the fuzzy set $B^{\downarrow_I} \in L^X$ of all objects having all the attributes of B is given by

$$B^{\downarrow_I}(x) = \bigwedge_{y \in Y} (B(y) \to I(x, y)). \tag{3}$$

In what follows we omit the subscript I if it is obvious. The direct formalization of Port-Royal ideas yields therefore the following definition: A *(formal)* **L**-*concept* (fuzzy concept) in a given fuzzy context $\langle X, Y, I\rangle$ is a pair $\langle A, B\rangle$ of $A \in L^X$ (extent) and $B \in L^Y$ (intent) such that $A^{\uparrow} = B$ and $B^{\downarrow} = A$. The set $\mathcal{B}(X, Y, I) = \{\langle A, B\rangle \in L^X \times L^Y \mid A^{\uparrow} = B, B^{\downarrow} = A\}$ of all fuzzy concepts is called an **L**-*concept lattice* (fuzzy concept lattice) given by the fuzzy context $\langle X, Y, I\rangle$. The term lattice is justified by the fact that the relation $\leq$ defined on $\mathcal{B}(X, Y, I)$ by

$$\langle A_1, B_1\rangle \leq \langle A_2, B_2\rangle \text{ iff } A_1 \subseteq A_2 \text{ (or iff } B_2 \subseteq B_1)$$

makes $\mathcal{B}(X, Y, I)$ into a complete lattice [4]. A moment reflection shows that $\leq$ models in a natural way the conceptual hierarchy, i.e. $\langle A_1, B_1\rangle \leq \langle A_2, B_2\rangle$ means that the concept $\langle A_2, B_2\rangle$ is more general than $\langle A_1, B_1\rangle$ ($\langle A_1, B_1\rangle$ is more specific than $\langle A_2, B_2\rangle$).

3. Reduction of fuzzy concept lattices to concept lattices

For $\mathbf{L} = \mathbf{2}$ (two-element Boolean algebra), the notions of **L**-concept and **L**-concept lattice coincide (modulo identifying **2**-sets with sets) with the notions of concept and concept lattice [17].

Our aim now is to show that each **L**-concept lattice can be viewed as a concept lattice. As a result, we obtain a simple proof of the theorem characterizing **L**-concept lattices. The proof makes a use of the Main theorem of concept lattices [17] the non-trivial part of which is the following assertion.

Proposition 3.1. *For a binary relation $I \subseteq X \times Y$, $\mathcal{B}(X, Y, I)$ is a complete lattice w.r.t. $\leq$. Moreover, an arbitrary complete lattice $\mathbf{V} = \langle V, \leq\rangle$ is isomorphic to $\mathcal{B}(X, Y, I)$ iff there are mappings $\gamma : X \to V$, $\mu : Y \to V$ such that $\gamma(X)$ is $\bigvee$-dense in $\mathbf{V}$, $\mu(Y)$ is $\bigwedge$-dense in $\mathbf{V}$, and $\gamma(x) \leq \mu(y)$ iff $\langle x, y\rangle \in I$.*

Note that $V' \subseteq V$ is $\bigvee$-dense ($\bigwedge$-dense) in V if each $v \in V$ is the supremum (infimum) of some subset of V'.

L-concept lattices may also be viewed as lattices of fixed points of **L**-Galois connections.

Definition 3.1. Let K be a $\leq$-filter. An $\mathbf{L}_K$-*Galois connection* (*fuzzy Galois connection*) between the sets X and Y is a pair $\langle {}^{\uparrow}, {}^{\downarrow}\rangle$ of mappings ${}^{\uparrow} : L^X \to L^Y$, ${}^{\downarrow} : L^Y \to L^X$, satisfying

$$\mathrm{S}(A_1, A_2) \leq \mathrm{S}(A_2^{\uparrow}, A_1^{\uparrow}) \quad \text{whenever } \mathrm{S}(A_1, A_2) \in K \tag{4}$$

$$\mathrm{S}(B_1, B_2) \leq \mathrm{S}(B_2^{\downarrow}, B_1^{\downarrow}) \quad \text{whenever } \mathrm{S}(B_1, B_2) \in K \tag{5}$$

$$A \subseteq (A^{\uparrow})^{\downarrow} \tag{6}$$

$$B \subseteq (B^{\downarrow})^{\uparrow} . \tag{7}$$

for every $A, A_1, A_2 \in L^X$, $B, B_1, B_2 \in L^Y$.

$\mathbf{L}_L$-Galois connections are called **L**-Galois connections. Note that Galois connections between sets [7] are just **2**-Galois connections. Given an $\mathbf{L}_K$-Galois connection $\langle {}^{\uparrow}, {}^{\downarrow}\rangle$ between X and Y, we call a fixed point of $\langle {}^{\uparrow}, {}^{\downarrow}\rangle$ each pair $\langle A, B\rangle$ of $A \in L^X$ and $B \in L^Y$ such that $A^{\uparrow} = B$ and $B^{\downarrow} = A$; furthermore, we denote by $\mathcal{B}\left(X, Y, \langle {}^{\uparrow}, {}^{\downarrow}\rangle\right)$ the set of all fixed points of $\langle {}^{\uparrow}, {}^{\downarrow}\rangle$, i.e.

$\mathcal{B}\left(X, Y, \langle^{\uparrow}, ^{\downarrow}\rangle\right) = \{\langle A, B\rangle \in L^X \times L^Y \mid A^{\uparrow} = B, B^{\downarrow} = A\}$. Since $\langle^{\uparrow}, ^{\downarrow}\rangle$ is a Galois connection between the complete lattices $\langle L^X, \subseteq\rangle$ and $\langle L^Y, \subseteq\rangle$ [7, 16], $\mathcal{B}\left(X, Y, \langle^{\uparrow}, ^{\downarrow}\rangle\right)$ is a complete lattice w.r.t. $\leq$ defined by $\langle A_1, B_1\rangle \leq \langle A_2, B_2\rangle$ iff $A_1 \subseteq A_2$ (iff $B_2 \subseteq B_1$), see e.g. [6].

Recall now the result obtained in [3] which implies that each **L**-concept lattice is in fact a lattice of fixed points of some **L**-Galois connection, and vice-versa.

Proposition 3.2. *For a binary* **L***-relation* $I \in L^{X\times Y}$ *denote* $\langle^{\uparrow_I}, ^{\downarrow_I}\rangle$ *the mappings defined for* $A \in L^X$, $B \in L^Y$ *by (2) and (3). For an* **L***-Galois connection* $\langle^{\uparrow}, ^{\downarrow}\rangle$ *between* X *and* Y *denote by* $I_{\langle\uparrow,\downarrow\rangle}$ *the binary* **L***-relation* $I \in L^{X\times Y}$ *defined for* $x \in X$, $y \in Y$ *by* $I(x, y) = \{\,^1/x\}^{\uparrow}(y)$ (= $\{\,^1/y\}^{\downarrow}(x)$)*. Then* $\langle^{\uparrow_I}, ^{\downarrow_I}\rangle$ *is an* **L***-Galois connection and it holds* $\langle^{\uparrow}, ^{\downarrow}\rangle = \langle^{\uparrow_{I_{\langle\uparrow,\downarrow\rangle}}}, ^{\downarrow_{I_{\langle\uparrow,\downarrow\rangle}}}\rangle$ *and* $I = I_{\langle\uparrow_I,\downarrow_I\rangle}$.

Since $1 \in K$ holds for each $\leq$-filter K, each $\mathbf{L}_K$-Galois connection is also an $\mathbf{L}_{\{1\}}$-Galois connection. We are going to show that $\mathbf{L}_{\{1\}}$-Galois connections between X and Y are in one-to-one correspondence with special **2**-Galois connections between $X \times L$ and $Y \times L$. Note that each **L**-set $A \in L^X$ is in fact a subset of $X \times L$, i.e. $A \subseteq X \times L$. However, the usual set-theoretical operations with **L**-sets defined componentwise (which is usual in fuzzy set theory) do not coincide with the operations defined on **L**-sets as on subsets of $X \times L$. In order to have such a correspondence, one may proceed as follows.

Definition 3.2. Call a subset $A \subseteq X \times L$ *(***L***-set)-representative* if (1) for each $x \in X$ it holds $\langle x, a\rangle \in A$ and $b \leq a$ implies $\langle x, b\rangle \in A$, and (2) for each $x \in X$ the set $\{a \in L \mid \langle x, a\rangle \in A\}$ has the greatest element.

For any **L**-set $A \in L^X$ put

$$\lfloor A\rfloor = \{\langle x, a\rangle \in X \times L \mid a \leq A(x)\}. \tag{8}$$

For any $A \subseteq X \times L$ put

$$\lceil A\rceil = \{\langle x, a\rangle \in X \times L \mid a = \bigvee_{\langle x,b\rangle\in A} b\}. \tag{9}$$

The following lemma is immediate.

Lemma 3.1. *Let* $A \in L^X$ *be an* **L***-set,* $A' \subseteq X\times L$ *be a representative set. Then (1)* $\lfloor A\rfloor \subseteq X\times L$ *is an representative set, (2)* $\lceil A'\rceil$ *is an* **L***-set such that (3)* $A = \lceil\lfloor A\rfloor\rceil$, $A' = \lfloor\lceil A'\rceil\rfloor$.

Definition 3.3. A **2**-Galois connection $\langle^{\wedge}, ^{\vee}\rangle$ between $X \times L$ and $Y \times L$ is called *commutative w.r.t.* $\lfloor\lceil\ \rceil\rfloor$ if

$$\lfloor\lceil A\rceil\rfloor^{\wedge} = \lfloor\lceil A^{\wedge}\rceil\rfloor \quad \text{and} \quad \lfloor\lceil B\rceil\rfloor^{\vee} = \lfloor\lceil B^{\vee}\rceil\rfloor \tag{10}$$

holds for each $A \in X \times L$, $B \in Y \times L$.

Remark 3.1. Note that $\lfloor\lceil A\rceil\rfloor^{\wedge} \subseteq A^{\wedge} \subseteq \lfloor\lceil A^{\wedge}\rceil\rfloor$ holds for any **2**-Galois connection $\langle^{\wedge},^{\vee}\rangle$ between $X \times L$ and $Y \times L$. Indeed, $\lfloor\lceil A\rceil\rfloor^{\wedge} \subseteq A^{\wedge}$ follows from $A \subseteq \lfloor\lceil A\rceil\rfloor$ and antitonicity of $^{\wedge}$, whereas $A^{\wedge} \subseteq \lfloor\lceil A^{\wedge}\rceil\rfloor$ follows from the fact that $A \subseteq \lfloor\lceil A\rceil\rfloor$ holds for any $A \in X \times L$. It follows that the first condition of (10) is equivalent to $\lfloor\lceil A\rceil\rfloor^{\wedge} \supseteq \lfloor\lceil A^{\wedge}\rceil\rfloor$. Moreover, in this case we have $\lfloor\lceil A\rceil\rfloor^{\wedge} = A^{\wedge} = \lfloor\lceil A^{\wedge}\rceil\rfloor$. Similarly for the second condition of (10) which is, in fact, equivalent to $\lfloor\lceil B\rceil\rfloor^{\vee} \supseteq \lfloor\lceil B^{\vee}\rceil\rfloor$.

For a pair $\langle^{\wedge},^{\vee}\rangle$ of mappings $^{\wedge} : X \times L \to Y \times L$, $^{\vee} : Y \times L \to X \times L$ define the pair $\langle^{\uparrow_{\langle\wedge,\vee\rangle}},^{\downarrow_{\langle\wedge,\vee\rangle}}\rangle$ of mappings $^{\uparrow_{\langle\wedge,\vee\rangle}} : L^X \to L^Y$, $^{\downarrow_{\langle\wedge,\vee\rangle}} : L^Y \to L^X$ by

$$A^{\uparrow_{\langle\wedge,\vee\rangle}} = \lceil\lfloor A\rfloor^{\wedge}\rceil \quad \text{and} \quad B^{\downarrow_{\langle\wedge,\vee\rangle}} = \lceil\lfloor B\rfloor^{\vee}\rceil \tag{11}$$

for $A \in L^X$, $B \in L^Y$. For a pair $\langle^{\uparrow},^{\downarrow}\rangle$ of mappings $^{\uparrow} : L^X \to L^Y$, $^{\downarrow} : L^Y \to L^X$ by define a pair $\langle^{\wedge_{\langle\uparrow,\downarrow\rangle}},^{\vee_{\langle\uparrow,\downarrow\rangle}}\rangle$ of mappings $^{\wedge_{\langle\uparrow,\downarrow\rangle}} : X \times L \to Y \times L$, $^{\vee_{\langle\uparrow,\downarrow\rangle}} : Y \times L \to X \times L$ by

$$A^{\wedge_{\langle\uparrow,\downarrow\rangle}} = \lfloor\lceil A\rceil^{\uparrow}\rfloor \quad \text{and} \quad B^{\vee_{\langle\uparrow,\downarrow\rangle}} = \lfloor\lceil B\rceil^{\downarrow}\rfloor \tag{12}$$

for $A \in X \times L$, $B \in Y \times L$.

Theorem 3.1. *Let $\langle^{\uparrow},^{\downarrow}\rangle$ be a $\mathbf{L}_{\{1\}}$-Galois connection between X and Y and $\langle^{\wedge},^{\vee}\rangle$ be a* **2***-Galois connection between $X \times L$ and $Y \times L$ which is commutative w.r.t. $\lfloor\lceil\ \rceil\rfloor$. Then the following conditions hold.*

(1) $\langle^{\wedge_{\langle\uparrow,\downarrow\rangle}},^{\vee_{\langle\uparrow,\downarrow\rangle}}\rangle$ is a **2***-Galois connection between $X \times L$ and $Y \times L$ which is commutative w.r.t. $\lfloor\lceil\ \rceil\rfloor$.*

(2) $\langle^{\uparrow_{\langle\wedge,\vee\rangle}},^{\downarrow_{\langle\wedge,\vee\rangle}}\rangle$ is a $\mathbf{L}_{\{1\}}$-Galois connection between X and Y.

(3) $\langle^{\wedge},^{\vee}\rangle = \langle^{\wedge_{\langle\uparrow_{\langle\wedge,\vee\rangle},\downarrow_{\langle\wedge,\vee\rangle}\rangle}},^{\vee_{\langle\uparrow_{\langle\wedge,\vee\rangle},\downarrow_{\langle\wedge,\vee\rangle}\rangle}}\rangle$ and $\langle^{\uparrow},^{\downarrow}\rangle = \langle^{\uparrow_{\langle\wedge_{\langle\uparrow,\downarrow\rangle},\vee_{\langle\uparrow,\downarrow\rangle}\rangle}},^{\downarrow_{\langle\wedge_{\langle\uparrow,\downarrow\rangle},\vee_{\langle\uparrow,\downarrow\rangle}\rangle}}\rangle$.

Proof:
(1) Let $A_1, A_2 \subseteq X \times L$, $A_1 \subseteq A_2$. We have $\mathrm{S}(\lceil A_1\rceil, \lceil A_2\rceil) = 1$, thus $\mathrm{S}(\lceil A_2\rceil^{\uparrow}, \lceil A_1\rceil^{\uparrow}) = 1$, and hence $A_2^{\wedge_{\langle\uparrow_{\langle\wedge,\vee\rangle},\downarrow_{\langle\wedge,\vee\rangle}\rangle}} = \lfloor\lceil A_2\rceil^{\uparrow}\rfloor \subseteq \lfloor\lceil A_1\rceil^{\uparrow}\rfloor = A_1^{\wedge_{\langle\uparrow_{\langle\wedge,\vee\rangle},\downarrow_{\langle\wedge,\vee\rangle}\rangle}}$. We have established that $A_1 \subseteq A_2$ implies $A_2^{\wedge_{\langle\uparrow_{\langle\wedge,\vee\rangle},\downarrow_{\langle\wedge,\vee\rangle}\rangle}} \subseteq A_1^{\wedge_{\langle\uparrow_{\langle\wedge,\vee\rangle},\downarrow_{\langle\wedge,\vee\rangle}\rangle}}$. Similarly for $B_1, B_2 \in Y \times L$.

For $A \subseteq X \times L$ we have $A^{\wedge_{\langle\uparrow_{\langle\wedge,\vee\rangle},\downarrow_{\langle\wedge,\vee\rangle}\rangle}\vee_{\langle\uparrow_{\langle\wedge,\vee\rangle},\downarrow_{\langle\wedge,\vee\rangle}\rangle}} = \lfloor\lceil\lfloor\lceil A\rceil^{\uparrow}\rfloor\rceil^{\downarrow}\rfloor = \lfloor\lceil A\rceil^{\uparrow\downarrow}\rfloor$. If $\langle x, a\rangle \in A$ then $\lceil A\rceil(x) \geq a$, thus $\lceil A\rceil^{\uparrow\downarrow}(x) \geq \lceil A\rceil(x) \geq a$, therefore $\langle x, a\rangle \in \lfloor\lceil A\rceil^{\uparrow\downarrow}\rfloor$, i.e. $A \subseteq A^{\vee_{\langle\uparrow_{\langle\wedge,\vee\rangle},\downarrow_{\langle\wedge,\vee\rangle}\rangle}\wedge_{\langle\uparrow_{\langle\wedge,\vee\rangle},\downarrow_{\langle\wedge,\vee\rangle}\rangle}}$. Similarly for $B \subseteq Y \times L$.

The commutativity follows by $\lfloor\lceil A\rceil\rfloor^{\wedge_{\langle\uparrow_{\langle\wedge,\vee\rangle},\downarrow_{\langle\wedge,\vee\rangle}\rangle}} = \lfloor\lceil\lfloor\lceil A\rceil\rfloor\rceil^{\uparrow}\rfloor = \lfloor\lceil A\rceil^{\uparrow}\rfloor = \lfloor\lceil\lfloor\lceil A\rceil^{\uparrow}\rfloor\rceil\rfloor = \lfloor\lceil A^{\wedge_{\langle\uparrow_{\langle\wedge,\vee\rangle},\downarrow_{\langle\wedge,\vee\rangle}\rangle}}\rceil\rfloor$.

(2) Let $A_1, A_2 \in L^X$, $\mathrm{S}(A_1, A_2) = 1$. Then $\lfloor A_1\rfloor \subseteq \lfloor A_2\rfloor$, $\lfloor A_2\rfloor^{\wedge} \subseteq \lfloor A_1\rfloor^{\wedge}$ and $A_2^{\uparrow_{\langle\wedge_{\langle\uparrow,\downarrow\rangle},\vee_{\langle\uparrow,\downarrow\rangle}\rangle}} = \lceil\lfloor A_2\rfloor^{\wedge}\rceil \subseteq \lceil\lfloor A_1\rfloor^{\wedge}\rceil = A_1^{\uparrow_{\langle\wedge_{\langle\uparrow,\downarrow\rangle},\vee_{\langle\uparrow,\downarrow\rangle}\rangle}}$. Similarly for $B_1, B_2 \in L^Y$.

For $A \in L^X$ we have $A^{\uparrow_{\langle \wedge_{\langle\uparrow,\downarrow\rangle}, \vee_{\langle\uparrow,\downarrow\rangle}\rangle} \downarrow_{\langle \wedge_{\langle\uparrow,\downarrow\rangle}, \vee_{\langle\uparrow,\downarrow\rangle}\rangle}} = \lceil\lfloor\lceil\lfloor A\rfloor^{\wedge}\rceil\rfloor^{\vee}\rceil = \lceil\lfloor\lceil\lfloor A\rfloor\rceil\rfloor^{\wedge\vee}\rceil = \lceil\lfloor A\rfloor^{\wedge\vee}\rceil \supseteq A$ due to the commutativity and the fact $\lfloor A\rfloor^{\wedge\vee} \supseteq \lfloor A\rfloor$. Similarly for $B \in L^Y$.

(3) Due to Remark following Definition 3.3 and the fact that for $A' \in L^X$ it holds $\lceil\lfloor A'\rfloor\rceil = A'$ we have $A^{\wedge_{\langle \uparrow_{\langle\wedge,\vee\rangle}, \downarrow_{\langle\wedge,\vee\rangle}\rangle}} = \lfloor\lceil A\rceil^{\uparrow_{\langle\wedge,\vee\rangle}}\rfloor = \lfloor\lceil\lfloor\lceil A\rceil\rfloor^{\wedge}\rceil\rfloor = \lfloor\lceil A^{\wedge}\rceil\rfloor = A^{\wedge}$ for any $A \subseteq X \times L$. For any $A \in L^X$, since $\lceil\lfloor A\rfloor\rceil = A$ holds, we have $A^{\uparrow_{\langle \wedge_{\langle\uparrow,\downarrow\rangle}, \vee_{\langle\uparrow,\downarrow\rangle}\rangle}} = \lceil\lfloor A\rfloor^{\wedge_{\langle\uparrow,\downarrow\rangle}}\rceil = \lfloor\lceil\lfloor\lceil A\rceil\rfloor^{\wedge}\rceil\rfloor = A^{\uparrow}$. Similarly for $B \subseteq Y \times L$ and $B \in L^Y$. □

Remark 3.2. The condition that $\langle^\wedge, ^\vee\rangle$ is commutative is essential in the foregoing statement. For consider $X = \{x\}$, $Y = \{y\}$, $L = \{0, \frac{1}{2}, 1\}$, and $I \subseteq (X \times L) \times (Y \times L)$ given by $I = \{\langle\langle x, 0\rangle, \langle y, \frac{1}{2}\rangle\rangle, \langle\langle x, \frac{1}{2}\rangle, \langle y, 0\rangle\rangle, \langle\langle x, \frac{1}{2}\rangle, \langle y, \frac{1}{2}\rangle\rangle, \langle\langle x, 1\rangle, \langle y, \frac{1}{2}\rangle\rangle, \langle\langle x, 1\rangle, \langle y, 1\rangle\rangle\}$. Let further $\langle^\wedge, ^\vee\rangle$ be the Galois connection between $X \times L$ and $Y \times L$ induced by I. Note that it is not commutative since the commutativity fails for $A = \{\langle x, 1\rangle\}$. For the mappings $\langle^\uparrow, ^\downarrow\rangle$ induced by $\langle^\wedge, ^\vee\rangle$ by (11) we have $\{\,^1/x\}^{\uparrow\downarrow} = \{\,^{\frac{1}{2}}/x\} \not\supseteq \{\,^1/x\}$, i.e. $\langle^\uparrow, ^\downarrow\rangle$ is not a $\mathbf{L}_{\{1\}}$-Galois connection.

Note also that the $\mathbf{L}_{\{1\}}$-Galois connection $\langle^\uparrow, ^\downarrow\rangle$ induced by a Galois connection $\langle^\wedge, ^\vee\rangle$ need not be an $\mathbf{L}$-Galois connection (a counterexample is easy to get).

Theorem 3.2. *For any $\mathbf{L}_K$-Galois connection $\langle^\uparrow, ^\downarrow\rangle$, $\mathcal{B}\left(X, Y, \langle^\uparrow, ^\downarrow\rangle\right)$ and $\mathcal{B}\left(X \times L, Y \times L, \langle^\wedge, ^\vee\rangle\right)$, where $\langle^\wedge, ^\vee\rangle = \langle^{\wedge_{\langle\uparrow,\downarrow\rangle}}, ^{\vee_{\langle\uparrow,\downarrow\rangle}}\rangle$ of Theorem 3.1, are isomorphic lattices. Moreover, $\mathcal{B}\left(X \times L, Y \times L, \langle^\wedge, ^\vee\rangle\right) = \mathcal{B}\left(X \times L, Y \times L, I^\times\right)$ where $\langle\langle x, \alpha\rangle, \langle y, \beta\rangle\rangle \in I^\times$ iff $\beta \leq \{\,^\alpha/x\}^\uparrow(y)$.*

Proof:
We prove the assertion by showing that $h : \mathcal{B}\left(X, Y, \langle^\uparrow, ^\downarrow\rangle\right) \to \mathcal{B}\left(X \times L, Y \times L, \langle^\wedge, ^\vee\rangle\right)$ and $g : \mathcal{B}\left(X \times L, Y \times L, \langle^\wedge, ^\vee\rangle\right) \to \mathcal{B}\left(X, Y, \langle^\uparrow, ^\downarrow\rangle\right)$ defined by $h(\langle A, B\rangle) = \langle\lfloor A\rfloor, \lfloor B\rfloor\rangle$ and $g(\langle A', B'\rangle) = \langle\lceil A'\rceil, \lceil B'\rceil\rangle$ are mutually inverse order-preserving maps. First, we show that h and g are correctly defined, i.e. $h(\mathcal{B}\left(X, Y, \langle^\uparrow, ^\downarrow\rangle\right)) \subseteq \mathcal{B}\left(X \times L, Y \times L, \langle^\wedge, ^\vee\rangle\right)$ and $g(\mathcal{B}\left(X \times L, Y \times L, \langle^\wedge, ^\vee\rangle\right)) \subseteq \mathcal{B}\left(X, Y, \langle^\uparrow, ^\downarrow\rangle\right)$. If $\langle A, B\rangle \in \mathcal{B}\left(X, Y, \langle^\uparrow, ^\downarrow\rangle\right)$ then $\lfloor A\rfloor^\wedge = \lfloor\lceil\lfloor A\rfloor\rceil^\uparrow\rfloor = \lfloor A^\uparrow\rfloor = \lfloor B\rfloor$ and, similarly, $\lfloor B\rfloor^\vee = \lfloor A\rfloor$, i.e. $h(\langle A, B\rangle) \in \mathcal{B}\left(X \times L, Y \times L, \langle^\wedge, ^\vee\rangle\right)$. If $\langle A', B'\rangle \in \mathcal{B}\left(X \times L, Y \times L, \langle^\wedge, ^\vee\rangle\right)$ then $\lceil A\rceil^\uparrow = \lceil\lfloor\lceil A'\rceil\rfloor^\wedge\rceil = \lceil A'^\wedge\rceil = \lceil B'\rceil$ and, similarly, $\lceil B'\rceil^\vee = \lceil A'\rceil$, i.e. $g(\langle A', B'\rangle) \in \mathcal{B}\left(X, Y, \langle^\uparrow, ^\downarrow\rangle\right)$. By Lemma 3.1, h and g are mutually inverse. Finally, both h and g are clearly order-preserving.

To see that $\mathcal{B}\left(X \times L, Y \times L, \langle^\wedge, ^\vee\rangle\right) = \mathcal{B}\left(X \times L, Y \times L, I^\times\right)$, it is enough to show that $I^\times$ is the relation $I_{\langle\wedge,\vee\rangle}$ corresponding to $\langle^\wedge, ^\vee\rangle$ by Proposition 3.2. This is, indeed, true, since $\langle\langle\alpha, x\rangle, \langle\beta, x\rangle\rangle \in I_{\langle\wedge,\vee\rangle}$ iff $\langle m, \beta\rangle \in \{\langle g, \alpha\rangle\}^\wedge$ which is by (12) equivalent to $\beta \leq \{\,^\alpha/x\}^\uparrow(y)$, i.e. to $\langle\langle\alpha, x\rangle, \langle\beta, x\rangle\rangle \in I^\times$. □

The following result shows that (modulo indentifying $\mathbf{L}$-sets with representative sets) $\mathbf{L}$-concept lattices may be viewed as concept lattices.

Theorem 3.3. *Any $\mathbf{L}$-concept lattice $\mathcal{B}\left(X, Y, I\right)$ is isomorphic to the concept lattice $\mathcal{B}\left(X \times L, Y \times L, I^\times\right)$ where $\langle\langle x, \alpha\rangle, \langle y, \beta\rangle\rangle \in I^\times$ iff $\alpha \otimes \beta \leq I(x, y)$.*

Proof:
Let $\langle^{\uparrow},^{\downarrow}\rangle$ be the **L**-Galois connection induced by I by Proposition 3.2. We have $\{{}^{\alpha}/x\}^{\uparrow}(y) = \bigwedge_{x'\in X}\{{}^{\alpha}/x\}(x') \to I(x,y) = \alpha \to I(x,y)$, i.e. the adjunction property gives $\beta \leq \{{}^{\alpha}/x\}^{\uparrow}(y)$ iff $\alpha\otimes\beta \leq I(x,y)$. The assertion now follows by By Theorem 3.2. □

The following theorem gives a characterization of lattices of fixed points of $\mathbf{L}_K$-Galois connections and, in particular, of **L**-concept lattices.

Theorem 3.4. *Let $\langle^{\uparrow},^{\downarrow}\rangle$ be an $\mathbf{L}_K$-Galois connection between X and Y. (1) The set $\mathcal{B}\left(X,Y,\langle^{\uparrow},^{\downarrow}\rangle\right)$ is under $\leq$ a complete lattice where the suprema and infima are given by*

$$\bigwedge_{j\in J}\langle A_j,B_j\rangle = \langle\bigcap_{j\in J}A_j,(\bigcup_{j\in J}B_j)^{\downarrow\uparrow}\rangle\ ,\ \bigvee_{j\in J}\langle A_j,B_j\rangle = \langle(\bigcup_{j\in J}A_j)^{\uparrow\downarrow},\bigcap_{j\in J}B_j\rangle\ . \tag{13}$$

(2) Moreover, an arbitrary complete lattice $\mathbf{V}=\langle V,\leq\rangle$ is isomorphic to $\mathcal{B}\left(X,Y,\langle^{\uparrow},^{\downarrow}\rangle\right)$ iff there are mappings $\gamma: X\times L\to V$, $\mu: Y\times L\to V$ such that

(i) $\gamma(X,L)$ is $\bigvee$-dense in V, $\mu(Y,L)$ is $\bigwedge$-dense in V;
(ii) $\gamma(x,\alpha)\leq\mu(y,\beta)$ iff $\beta\leq\{{}^{\alpha}/x\}^{\uparrow}(y)$.

*If $\langle^{\uparrow},^{\downarrow}\rangle$ is an **L**-Galois connection, i.e. $\mathcal{B}\left(X,Y,\langle^{\uparrow},^{\downarrow}\rangle\right)=\mathcal{B}(X,Y,I)$ for some $I\in L^{X\times Y}$, then (ii) may be replaced by (ii') $\gamma(x,\alpha)\leq\mu(y,\beta)$ iff $\alpha\otimes\beta\leq I(x,y)$.*

Proof:
(1) follows from the description of infima and suprema of fixed points of Galois connection between complete lattices (see e.g. [16]) and the fact that $\langle^{\uparrow},^{\downarrow}\rangle$ is a Galois connection between the complete lattices $\langle L^X,\subseteq\rangle$ and $\langle L^Y,\subseteq\rangle$. (2) Follows directly by Proposition 3.1, Theorem 3.2, and Theorem 3.3. □

Remark 3.3. (1) Note that the characterization of **L**-concept lattices contained in Theorem 3.4 has been obtained in [4] by a direct proof without the reference to Proposition 3.1. Theorem 3.3, however, makes the characterization a direct consequence of Proposition 3.1.

(2) Let us also remark that by [17], each complete lattice $\langle V,\leq\rangle$ is isomorphic to the crisp concept lattice $\mathcal{B}(V,V,\leq)$. Therefore, any **L**-concept lattice $\mathcal{B}(X,Y,I)$ is isomorphic to the crisp concept lattice $\mathcal{B}(\mathcal{B}(X,Y,I),\mathcal{B}(X,Y,I),\leq)$. However, such a representation is unnatural compared to that one of provided by Theorem 3.3. Moreover, our representation yields almost directly Theorem 3.4, i.e. the characterization theorem for **L**-concept lattices.

Remark 3.4. Note also that formulas (13) can be derived directly from the assumption that they are valid for $\mathbf{L}=\mathbf{2}$. Namely, since, by Theorem 3.2, $\mathcal{B}\left(X,Y,\langle^{\uparrow},^{\downarrow}\rangle\right)$ and $\mathcal{B}(X\times L,Y\times L,\langle^{\wedge},^{\vee}\rangle)$ are isomorphic lattices with h,g (of Proof of Theorem 3.2) being the mutually inverse isomorphisms, it holds $\bigwedge_{j\in J}\langle A_j,B_j\rangle = g(h(\bigwedge_{j\in J}\langle A_j,B_j\rangle))$. Furthermore, we have

$$g(h(\bigwedge_{j\in J}\langle A_j,B_j\rangle)) = g(\bigwedge_{j\in J}h(\langle A_j,B_j\rangle)) =$$

$$
\begin{aligned}
&= g(\bigwedge_{j\in J}\langle\lfloor A_j\rfloor,\lfloor B_j\rfloor\rangle) = g(\langle\bigcap_{j\in J}\lfloor A_j\rfloor,(\bigcup_{j\in J}\lfloor B_j\rfloor)^{\vee\wedge}\rangle) = \\
&= \langle\lceil\bigcap_{j\in J}\lfloor A_j\rfloor\rceil,\lceil(\bigcup_{j\in J}\lfloor B_j\rfloor)^{\vee\wedge}\rceil\rangle = \langle\bigcap_{j\in J}\lceil\lfloor A_j\rfloor\rceil,\lceil\lfloor\lceil\bigcup_{j\in J}\lfloor B_j\rfloor\rceil^{\downarrow}\rfloor^{\wedge}\rceil\rangle = \\
&= \langle\bigcap_{j\in J}A_j,\lceil\lfloor\lceil\lfloor\lceil\bigcup_{j\in J}\lfloor B_j\rfloor\rceil^{\downarrow}\rfloor\rceil^{\uparrow}\rfloor\rceil\rangle = \langle\bigcap_{j\in J}A_j,\lceil\bigcup_{j\in J}\lfloor B_j\rfloor\rceil^{\downarrow\uparrow}\rangle = \\
&= \langle\bigcap_{j\in J}A_j,\bigcup_{j\in J}\lceil\lfloor B_j\rfloor\rceil^{\downarrow\uparrow}\rangle = \langle\bigcap_{j\in J}A_j,(\bigcup_{j\in J}B_j)^{\downarrow\uparrow}\rangle,
\end{aligned}
$$

i.e. the first part of (13) is valid. We used the validity of (13) for $\mathbf{L} = \mathbf{2}$, Lemma 3.1, the equalities $\lceil\bigcap_{j\in J}\lfloor C_j\rfloor\rceil = \bigcap_{j\in J}\lceil\lfloor C_j\rfloor\rceil$, $\lceil\bigcup_{j\in J}\lfloor C_j\rfloor\rceil = \bigcup_{j\in J}\lceil\lfloor C_j\rfloor\rceil$ (which hold for $\mathbf{L}$-sets C_j), and the definitions of $^\wedge$ and $^\vee$. The second part of (13) can be obtained dually.

4. Conclusion

We have shown that each **L**-concept lattice can be viewed as a concept lattice. Our results are more general in that they concern lattices of fixed points of $\mathbf{L}_K$-Galois connections of which **L**-concept lattices are a special case. The **L**-concept lattice is interpreted as a structure of concepts determined by a given **L**-context, i.e. a set of objects, a set of attributes, and an **L**-relation "to have" between objects and attributes. In this respect, the context of the corresponding concept lattice has no clear interpretation - each object of the new context is a pair $\langle x, \alpha\rangle$ where x is an object of the original **L**-context and α is a truth value, similarly for attributes. However, the result enables us to apply the results obtained for concept lattices to **L**-concept lattices (a problem concerning an **L**-concept lattice is to be "translated" into a problem concerning the corresponding concept lattice, the translated problem is to be solved by results available for concept lattices, and the solution obtained is to be translated back). In this way we obtained a simple proof of the theorem characterizing **L**-concept lattices (obtained originally directly in [4]). On the other hand, there are several phenomena which are degenerate in the case of (**2**-)concept lattices (e.g. similarity [5], logical precision [6] etc.). Clearly, the study of such phenomena cannot be "reduced" in the above mentioned way.

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