Similarity relations and BK-relational products [1]

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Abstract. (BK)-relational products and their role in the analysis and synthesis of complex systems has been discussed by Bandler and Kohout in a series of papers. We show that BK-products preserve similarity of the relations in that if R_1, R_2, and S_1, S_2 are similar then $R_1 * S_1$ and $R_2 * S_2$ are also similar for any type $*$ of BK-products. Applications of the results are discussed.

Key words: fuzzy relation, residuated lattice, relational product, similarity

1 Introduction and preliminaries

Let R and S be binary relations between X and Y, and Y and Z, respectively. A standard construction yields from R and S a new relation $R \circ S$ between X and Z relating $x \in X$ and $z \in Z$ whenever there is some $y \in Y$ such that both x and y are related by R and y and z are related by S. As discussed by Bandler and Kohout (see e.g. [1, 2, 3]) there are more ways to define a product (composition) of the relations R and S. These products (referred to as BK-products) have been shown useful for the analysis and synthesis of complex systems. Following the authors, we distinguish *circlet* product $\circ$, *subproduct* $\lhd$, *superproduct* $\rhd$, and *square* product $\Box$. Any of these products (denote it in general by $*$) yields a new relation $R * S$ from R and S. Using the authors' example and interpreting X, Y, and Z as the set of patients, symptoms of diseases, and diseases, respectively, the meaning of the products is as follows (uTv means that u and v are related by T): xRy iff patient x has the symptom y, ySz iff y is one of the symptoms characterizing disease z, $xR \circ Sz$ iff there is a symptom y of z such that x displays z, $xR \lhd Sz$ iff every symptom of x is one of those characterizing z, $xR \rhd Sz$ iff x has all the symptoms of z, $xR \Box Sz$ iff x has exactly those symptoms characterizing z. In the following we assume that R and S are fuzzy relations. The definitions of the products will be given in Section 2.

Recall that a binary fuzzy relation R between the sets X and Y is a mapping $R : X \times Y \to L$, where L is a support of some structure $\mathbf{L}$ of truth values. The

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value $R(x,y) \in L$ is interpreted as the truth value of the fact that x and y are related (by the relation represented by R). It has been argued (see e.g. [6, 7]) that from the logical point of view, $\mathbf{L}$ should obey at least the structure of a complete residuated lattice. Residuated lattices play nowadays a prominent role of structures of truth values of various fuzzy logical calculi [8, 9, 11].

Definition 1 *A* complete residuated lattice *is an algebra* $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ *such that*

> $\langle L, \wedge, \vee, 0, 1\rangle$ *is a complete lattice with the least element* 0 *and the greatest element* 1*,*
>
> $\langle L, \otimes, 1\rangle$ *is a commutative monoid, i.e.* $\otimes$ *is commutative, associative, and* $x \otimes 1 = x$ *holds holds for each* $x \in L$*, and*
>
> $\otimes, \rightarrow$ *form an adjoint pair, i.e.*
>
> $$x \otimes y \leq z \text{ iff } x \leq y \rightarrow z \qquad (1)$$
>
> *holds for all* $x, y, z \in L$*.*

Residuated lattices were introduced in [13]. In each residuated lattice it holds that $x \leq y$ implies $x \otimes z \leq y \otimes z$ (isotonicity), and $x \leq y$ implies $z \rightarrow x \leq z \rightarrow y$ (isotonicity in the second argument) and $x \rightarrow z \geq y \rightarrow z$ (antitonicity in the first argument). The operation $\otimes$ is thus a *t-norm* (see e.g. [8, 10]), $\rightarrow$ is called *residuum*. In the following we will deal with *complete residuated lattices*, i.e. $\langle L, \wedge, \vee, 0, 1\rangle$ is assumed to be a complete lattice.

A semantically complete first-order many-valued logic with semantics defined over complete residuated lattices can be found in [9]. Several special classes of residuated lattices serve as special structures of semantically complete logical calculi (for details see [8]).

The most studied and applied set of truth values is the real interval $[0,1]$ with $a \wedge b = \min(a,b)$, $a \vee b = \max(a,b)$, and with three important pairs of adjoint operations: the Łukasiewicz one ($a \otimes b = \max(a+b-1, 0)$, $a \rightarrow b = \min(1-a+b, 1)$), Gödel one ($a \otimes b = \min(a,b)$, $a \rightarrow b = 1$ if $a \leq b$ and $= b$ else), and product one ($a \otimes b = a \cdot b$, $a \rightarrow b = 1$ if $a \leq b$ and $= b/a$ else). For the role of these "building blocks" in fuzzy logic see [8]. Another important set of truth values is the set (the ordering determines the complete lattice structure) $\{a_0 = 0, a_1, \ldots, a_n = 1\}$ ($a_0 < \cdots < a_n$). Two t-norms are often considered: $a_k \otimes a_l = a_{\max(k+l-n,0)}$ and the corresponding $\rightarrow$ given by $a_k \rightarrow a_l = a_{\min(n-k+l,n)}$ (Łukasiewicz) and $a_k \otimes a_l = a_{\min(k,l)}$ and the corresponding $\rightarrow$ given by $a_k \rightarrow a_l = 1$ if $k \leq l$ and a_l otherwise (Gödel). A special case of the latter algebras is the Boolean algebra $\mathbf{2}$ of classical logic with the support $2 = \{0,1\}$. It may be easily verified that the only t-norm on $\{0,1\}$ is the classical conjunction operation $\wedge$, i.e. $a \wedge b = 1$ iff $a = 1$ and $b = 1$, which implies that the only residuum operation is the classical implication operation $\rightarrow$, i.e. $a \rightarrow b = 0$ iff $a = 1$ and $b = 0$. Note that each of the preceding residuated lattices is complete.

It can be shown that in each complete residuated lattice the identity

$$a \otimes \Big(\bigvee_{j\in J} b_j\Big) = \bigvee_{j\in J} (a \otimes b_j) \tag{2}$$

holds.

The t-norm $\otimes$ and the residuum $\rightarrow$ are intended for modeling of the conjunction and implication, respectively. Supremum ($\bigvee$) and infimum ($\bigwedge$) are intended for modeling of general and existential quantifier, respectively.

The following identities of complete residuated lattices will be needed:

$$a \otimes (a \rightarrow b) \leq b \tag{3}$$

$$a \otimes \bigwedge_{j\in J} b_j \leq \bigwedge_{j\in J} (a \otimes b_j) \tag{4}$$

Throughout this text we suppose that **L** is a complete residuated lattice. To stress explicitly the structure **L**, one sometimes speaks about **L**-relations instead of fuzzy relations etc.

2 BK-products and similarity of relations

A crucial role in the way humans regard the world is played by the *indiscernibility phenomenon* and the related *similarity phenomenon*. In fuzzy set theory, similarity phenomenon is approached via the so called similarity relations. By a $\otimes$*-similarity relation* (or fuzzy $\otimes$-equivalence relation, L-valued global equality, or simply similarity relation) [9, 10, 14] on a universe U it is meant a binary fuzzy relation E satisfying the following properties for all $x, y, z \in U$:

$$E(x,x) = 1 \tag{5}$$

$$E(x,y) = E(y,x) \tag{6}$$

$$E(x,y) \otimes E(y,z) \leq E(x,z). \tag{7}$$

Properties (5), (6), and (7) are called reflexivity, symmetricity, and transitivity, respectively. It is easily seen that in the crisp case, i.e. $L = \{0,1\}$, similarity relations are equivalence relations.

To be able to model the equivalence of truth values the so called *biresiduum* (or biimplication) [9, 11] operation $\leftrightarrow$ is defined by

$$a \leftrightarrow b = (a \rightarrow b) \wedge (b \rightarrow a).$$

The following lemma will be useful in the following considerations (see e.g. [4, 5]).

Lemma 2 *Let $\mathcal{S} = \{A_i \in L^U \mid i \in I\}$ be a family of fuzzy sets. The relation $E_{\mathcal{S}}$ defined by*

$$E_{\mathcal{S}}(x,y) = \bigwedge_{i\in I} (A_i(x) \leftrightarrow A_i(y)) \tag{8}$$

is a $\otimes$-similarity relation on U.

Notice that for the crisp case (i.e. $L = \{0, 1\}$), $E_{\mathcal{S}}$ is a crisp equivalence relation—two elements of the universe are equivalent iff there is no set of the family which separates them.

In the following we define a similarity relation on the set of all binary fuzzy relations and show that for each type $*$ of BK-products it holds that

if R_1, R_2 are similar and S_1, S_2 are similar then $R_1 * S_1$, $R_2 * S_2$ are similar.

Note that these considerations may be regarded as a kind of an analysis of sensitivity of the relationally-based model. For instance, returning to the patient–symptom–disease example, one may consider the relation S as relatively fixed (the values $S(y, z)$ are based on a long-term experience) while the relation R may be subject to error (in some degree). If we denote by R' the "right" (i.e. error-free) relation, then the similarity of R and R' may be thought of as the degree of correctness of the information carried by R. The above result (see the theorems below for precise formulation) then asserts that the degree of correctness of the information given by $R * S$ is at least as high as that one of the information given by R.

Lemma 3 *Let V_1 and V_2 be sets. The relation E defined by*

$$E(T_1, T_2) = \bigwedge_{\langle v_1, v_2\rangle \in V_1 \times V_2} T_1(v_1, v_2) \leftrightarrow T_2(v_1, v_2)$$

is a similarity relation on the set $L^{V_1 \times V_2}$ of all $\mathbf{L}$-relations between V_1 and V_2.

Proof. Directly by applying Lemma 2: put $I =_{\mathrm{def}} V_1 \times V_2$, $U =_{\mathrm{def}} L^{V_1 \times V_2}$ and $A_i(x) =_{\mathrm{def}} x(i)$ for $x \in L^{V_1 \times V_2}$, $i \in V_1 \times V_2$. $\square$

Lemma 3 gives thus a natural way to define a similarity degree of two relations. Note that on the language level, the similarity degree $E(T_1, T_2)$ is the truth value of the fact that for all pairs $\langle v_1, v_2\rangle$ it holds that v_1, v_2 are related by T_1 iff they are related by T_2.

Let $R \in L^{X \times Y}$, $S \in L^{Y \times Z}$ are binary $\mathbf{L}$-relations. The BK-products (see [1, 2]) of R and S are the $\mathbf{L}$-relations of $L^{X \times Z}$ defined by

$$\begin{aligned}
(R \circ S)(x, z) &= \bigvee_{y \in Y} (R(x, y) \otimes S(y, z)) \\
(R \lhd S)(x, z) &= \bigwedge_{y \in Y} (R(x, y) \to S(y, z)) \\
(R \rhd S)(x, z) &= \bigwedge_{y \in Y} (S(y, z) \to R(x, y)) \\
(R \,\square\, S)(x, z) &= \bigwedge_{y \in Y} (R(x, y) \leftrightarrow S(y, z))
\end{aligned}$$

for all $x \in X$, $z \in Z$.

Definition 4 *Let $*$ be one of the above defined BK-products. We say that $*$* preserves similarity *iff*

$$E(R_1, R_2) \otimes E(S_1, S_2) \leq E(R_1 * S_1, R_2 * S_2)$$

holds for every $R_1, R_2 \in L^{X \times Y}$, $S_1, S_2 \in L^{Y \times Z}$.

Note that Definition 4 captures just the preservation of similarity discussed above. To show the announced result we need the following lemmata.

Lemma 5 *For any $R, R_1, R_2 \in L^{X \times Y}$, $S, S_1, S_2 \in L^{Y \times Z}$ it holds $E(R_1, R_2) \leq E(R_1 \circ S, R_2 \circ S)$ and $E(S_1, S_2) \leq E(R \circ S_1, R \circ S_2)$.*

Proof. We prove only $E(R_1, R_2) \leq E(R_1 \circ S, R_2 \circ S)$. The second part is symmetric. We have to show

$$E(R_1, R_2) \leq \bigwedge_{\langle x,z \rangle \in X \times Z} (R_1 \circ S)(x, z) \leftrightarrow (R_2 \circ S)(x, z)$$

which holds iff for each $x \in X, z \in Z$ it holds

$$\begin{aligned} E(R_1, R_2) &\leq (R_1 \circ S)(x, z) \leftrightarrow (R_2 \circ S)(x, z) = \\ &= ((R_1 \circ S)(x, z) \to (R_2 \circ S)(x, z)) \wedge ((R_2 \circ S)(x, z) \to (R_1 \circ S)(x, z)) \end{aligned}$$

which holds iff both

$$E(R_1, R_2) \leq (R_1 \circ S)(x, z) \to (R_2 \circ S)(x, z)$$

and

$$E(R_1, R_2) \leq (R_2 \circ S)(x, z) \to (R_1 \circ S)(x, z)$$

hold. Due to symmetry we prove only the first inequality which holds iff

$$E(R_1, R_2) \otimes (R_1 \circ S)(x, z) \leq (R_2 \circ S)(x, z),$$

i.e.

$$E(R_1, R_2) \otimes \bigvee_{y \in Y} (R_1(x, y) \otimes S(y, z)) \leq \bigvee_{y \in Y} (R_2(x, y) \otimes S(y, z)).$$

We have

$$\begin{aligned} &E(R_1, R_2) \otimes \bigvee_{y \in Y} (R_1(x, y) \otimes S(y, z)) = \bigvee_{y \in Y} (E(R_1, R_2) \otimes R_1(x, y) \otimes S(y, z)) = \\ &= \bigvee_{y \in Y} (R_1(x, y) \otimes \bigwedge_{\langle x', y' \rangle \in X \times Y} (R_1(x', y') \leftrightarrow R_2(x', y')) \otimes S(y, z)) \leq \\ &\leq \bigvee_{y \in Y} (R_1(x, y) \otimes (R_1(x, y) \to R_2(x, y)) \otimes S(y, z)) \leq \bigvee_{y \in Y} (R_2(x, y) \otimes S(y, z)) \end{aligned}$$

which had to be proved. □

Lemma 6 *For any $R, R_1, R_2 \in L^{X\times Y}$, $S, S_1, S_2 \in L^{Y\times Z}$ it holds $E(R_1, R_2) \leq E(R_1 \lhd S, R_2 \lhd S)$ and $E(S_1, S_2) \leq E(R \lhd S_1, R \lhd S_2)$.*

Proof. Prove first $E(R_1, R_2) \leq E(R_1 \lhd S, R_2 \lhd S)$. This is equivalent to

$$E(R_1, R_2) \leq \bigwedge_{\langle x,z\rangle \in X\times Z} ((R_1 \lhd S)(x, z) \leftrightarrow (R_2 \lhd S)(x, z))$$

which holds iff for each $x \in X, z \in Z$

$$E(R_1, R_2) \leq ((R_1 \lhd S)(x, z) \leftrightarrow (R_2 \lhd S)(x, z))$$

which holds iff both

$$E(R_1, R_2) \leq ((R_1 \lhd S)(x, z) \rightarrow (R_2 \lhd S)(x, z))$$

and

$$E(R_1, R_2) \leq ((R_2 \lhd S)(x, z) \rightarrow (R_1 \lhd S)(x, z))$$

holds. We prove only the first part (the second one is symmetric) which holds iff

$$E(R_1, R_2) \otimes (R_1 \lhd S)(x, z) \leq \bigwedge_{y\in Y} (R_2(x, y) \rightarrow S(y, z))$$

iff for each $y \in Y$

$$E(R_1, R_2) \otimes (R_1 \lhd S)(x, z) \leq R_2(x, y) \rightarrow S(y, z)$$

iff

$$R_2(x, y) \otimes E(R_1, R_2) \otimes (R_1 \lhd S)(x, z) \leq S(y, z). \tag{9}$$

We have

$$\begin{aligned}
&R_2(x, y) \otimes E(R_1, R_2) \otimes (R_1 \lhd S)(x, z) = \\
&= R_2(x, y) \otimes \bigwedge_{\langle x',y'\rangle \in X\times Y} (R_1(x', y') \leftrightarrow R_2(x', y')) \otimes \\
&\quad \otimes \bigwedge_{y'\in Y} (R_1(x, y') \rightarrow S(y', z)) \leq \\
&\leq R_2(x, y) \otimes (R_2(x, y) \leftrightarrow R_1(x, y)) \otimes (R_1(x, y) \rightarrow S(y, z)) \leq S(y, z),
\end{aligned}$$

i.e. (9) holds.

Prove $E(S_1, S_2) \leq E(R \lhd S_1, R \lhd S_2)$. The inequality is equivalent to

$$E(S_1, S_2) \leq \bigwedge_{\langle x,z\rangle \in X\times Z} ((R \lhd S_1)(x, z) \leftrightarrow (R \lhd S_2)(x, z))$$

which holds iff

$$E(S_1, S_2) \leq ((R \lhd S_1)(x, z) \leftrightarrow (R \lhd S_2)(x, z))$$

holds for every $x \in X, z \in Z$ which holds iff both

$$E(S_1, S_2) \leq ((R \lhd S_1)(x,z) \to (R \lhd S_2)(x,z))$$

and

$$E(S_1, S_2) \leq ((R \lhd S_2)(x,z) \to (R \lhd S_1)(x,z))$$

holds. We again prove only the first part. It holds iff

$$(R \lhd S_1)(x,z) \otimes E(S_1, S_2) \leq \bigwedge_{y \in Y} (R(x,y) \to S_2(y,z))$$

iff for each $y \in Y$

$$(R \lhd S_1)(x,z) \otimes E(S_1, S_2) \leq R(x,y) \to S_2(y,z)$$

iff

$$R(x,y) \otimes (R \lhd S_1)(x,z) \otimes E(S_1, S_2) \leq S_2(y,z)$$

We have

$$\begin{aligned} & R(x,y) \otimes (R \lhd S_1)(x,z) \otimes E(S_1, S_2) \leq \\ = \quad & R(x,y) \otimes \bigwedge_{y' \in Y} (R(x,y') \to S_1(y',z)) \otimes \\ & \otimes \bigwedge_{\langle y',z' \rangle \in Y \times Z} (S_1(y',z') \leftrightarrow S_2(y',z')) \leq \\ \leq \quad & R(x,y) \otimes (R(x,y) \to S_1(y,z)) \otimes (S_1(y,z) \leftrightarrow S_2(y,z)) \leq S_2(y,z) \end{aligned}$$

which had to be proved. The lemma is proved. □

Lemma 7 *For any $R, R_1, R_2 \in L^{X \times Y}$, $S, S_1, S_2 \in L^{Y \times Z}$ it holds $E(R_1, R_2) \leq E(R_1 \rhd S, R_2 \rhd S)$ and $E(S_1, S_2) \leq E(R \rhd S_1, R \rhd S_2)$.*

Proof. The proof is analogous to the proof of Lemma 6 and hence omitted. □

Lemma 8 *For any $R, R_1, R_2 \in L^{X \times Y}$, $S, S_1, S_2 \in L^{Y \times Z}$ it holds $E(R_1, R_2) \leq E(R_1 \square S, R_2 \square S)$ and $E(S_1, S_2) \leq E(R \square S_1, R \square S_2)$.*

Proof. Due to symmetricity we prove only $E(R_1, R_2) \leq E(R_1 \square S, R_2 \square S)$ which is equivalent to

$$E(R_1, R_2) \leq \bigwedge_{\langle x,z \rangle \in X \times Z} ((R_1 \square S)(x,z) \leftrightarrow (R_2 \square S)(x,z))$$

which holds iff for every $x \in X, z \in Z$

$$E(R_1, R_2) \leq ((R_1 \square S)(x,z) \leftrightarrow (R_2 \square S)(x,z))$$

which holds iff both

$$E(R_1, R_2) \leq ((R_1 \square S)(x,z) \rightarrow (R_2 \square S)(x,z))$$

and

$$E(R_1, R_2) \leq ((R_2 \square S)(x,z) \rightarrow (R_1 \square S)(x,z))$$

holds. We prove only the first part which holds iff

$$E(R_1, R_2) \otimes ((R_1 \square S)(x,z) \leq \bigwedge_{y \in Y} (R_2(x,y) \leftrightarrow S(y,z))$$

which holds iff each of the iff for each $y \in Y$

$$E(R_1, R_2) \otimes ((R_1 \square S)(x,z) \leq (R_2(x,y) \rightarrow S(y,z)) \wedge (S(y,z) \rightarrow R_2(x,y)).$$

We check only the first of the required inequalities, i.e.

$$E(R_1, R_2) \otimes ((R_1 \square S)(x,z) \leq R_2(x,y) \rightarrow S(y,z)$$

which is equivalent to

$$R_2(x,y) \rightarrow E(R_1, R_2) \otimes ((R_1 \square S)(x,z) \leq S(y,z). \tag{10}$$

Since $(R_1 \square S)(x,z) \leq (R_1 \lhd S)(x,z)$, (10) follows by the last series of inequalities of the proof of Lemma 6. $\square$

Theorem 9 *Any of the BK-products preserves similarity.*

Proof. Let $*$ be a BK-product. By Lemmata 5, 6, 7, 8, and by the fact that E is a similarity relation and satisfies therefore (7) we have $E(R_1, R_2) \otimes E(S_1, S_2) \leq E(R_1 * S_1, R_2 * S_1) \otimes E(R_2 * S_1, R_2 * S_2) \leq E(R_1 * S_1, R_2 * S_2)$, i.e. $*$ preserves similarity. $\square$

Note that the basic procedure of approximate deduction, i.e. the Zadeh's compositional rule of inference [10] is a special case of BK-products. Namely, putting $X = \{x\}$, a one element set, the relation R can be interpreted as a fuzzy set A in Y (put $A(y) =_{\text{def}} R(x,y)$). A represents the input (observed phenomenon) to the deduction. S then represents the rule base. $A \circ S$ is then the output fuzzy set obtained by the compositional rule of inference. Our results say that (1) given a rule base represented by S, similar inputs are mapped to similar outputs, and (2) given an input A, similar rule bases map this input to similar outputs.

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