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Radim Belohlavek

The fourth inequality from the end of the long chain of inequalities in the proof of Lemma 3.22 (Lemma 24 in the preprint) is wrong. Hence Lemma 3.22 and also Theorem 3.23 (Theorem 25 in the preprint) are wrong.

A counterexample: Take $G = \{g\}$, $M = \{m, n\}$, and two crisp formal contexts, I_1 and I_2, given by $I_1(x, y) = 0$ and $I_2(x, y) = 0$ for all $x \in G$ and $y \in M$ except for $I_2(g, m) = 1$. Then $E(I_1, I_2) = 0$ but $E^{Ext}(\mathcal{B}_1, \mathcal{B}_2) = 1$ because both $\mathcal{B}_1 = \mathcal{B}(G, M, I_1)$ and $\mathcal{B}_2 = \mathcal{B}(G, M, I_2)$ have the same set of extents, namely $\{\emptyset, \{g\}\}$.

Similarity Relations in Concept Lattices

RADIM BĚLOHLÁVEK, *Institute for Research and Applications of Fuzzy Modeling, University of Ostrava, Bráfova 7, 701 03 Ostrava, Czech Republic.*
and Department of Computer Science, Technical University of Ostrava, tř. 17. listopadu, 708 33 Ostrava-Poruba, Czech Republic.
E-mail: radim.belohlavek@osu.cz

Abstract

This paper studies the issue of similarity relations in fuzzy concept lattices. Fuzzy concepts and fuzzy concept lattices represent a formal approach to the modelling of non-sharp (fuzzy) concepts and conceptual structures in the sense of traditional (Port-Royal) logic. Applications of concept lattices are in representation of conceptual knowledge and in conceptual analysis of (fuzzy) data. Similarity relations are defined and considered on three levels: similarity of objects (and similarity of attributes), similarity of concepts, and similarity of concept lattices. We show a way to factorize (simplify) concept lattices by the similarity of concepts. Also shown is how to reduce the computation of the similarity relations.

Keywords: Concept, similarity, fuzzy logic, residuated lattice, concept lattice, tolerance.

1 Introduction

Human thinking is often identified with reasoning with concepts. The fundamental role of concepts in reasoning is reflected in their pivotal role in the early development of logic. The learning on concepts constitutes one of the three parts of traditional (or Port-Royal) logic [1]. On the intuitive level, the formation of human concepts (like MAMMAL, RED APPLE etc.) is a typical example of what is meant by *information granulation* [21]. Concepts are thought of as granules of information which (on a higher level of abstraction) can be taken as units for reasoning.

A formal theory of concepts and conceptual structures (so called concept lattices) inspired by the traditional logic has been initiated by Wille in [17]. The theory of concept lattices is now a well-elaborated one with direct applications in conceptual data analysis and conceptual knowledge representation (see [7] or the survey [19]). A generalization of this theory (Wille's theory is, in fact, a theory of sharp (i.e. two-valued) concepts) from the point of view of fuzzy logic (graded truth approach [10, 11]) has been presented in [2, 3, 4]. The aim of this paper is to study the similarity phenomenon in fuzzy conceptual structures. The motivation of our study is twofold: first, the study of similarities (on various levels) is apparently natural in the context of conceptual structures and, second, it is similarity due to which a reduction of complexity (of conceptual structures, in our case) can be attained (cf. the principle of incompatibility [13, pp. 329–330]).

The organization of the paper is as follows. In Section 2 we recall the fundamental notions and results. Section 3 is devoted to the study of similarity relations. Section 4 contains an illustrative example.

2 Fuzzy concepts and concept lattices

By traditional (or Port-Royal) logic [1, 15]) a *concept* is determined by its *extent*, a collection of all objects covered by the concept, and its *intent*, a collection of attributes (properties) covered by the concept. Thus, the extent of the concept DOG consists of all dogs (or living dogs at a fixed time at a fixed possible world to avoid the philosophical problems) while the intent of DOG consists of all attributes of all dogs (i.e. 'to bark', 'to be a mammal', 'to have four extremities' etc.). The primary relation appearing on concepts is that of hierarchical ordering. For instance, the concepts DOG and CAT are not comparable w.r.t. to conceptual hierarchy, while the concept MAMMAL is more abstract (higher in the hierarchical ordering) than both DOG and CAT. A characteristic feature of empirical concepts (e.g. BIG DOG) is that the extent and the intent are non-sharp (fuzzy) collections (some dogs are bigger than others, various dogs belong to the extent to various (not only two, i.e. fully yes or fully no) degrees). In [2, 3, 4] a theory of fuzzy concepts and hierarchical structures of fuzzy concepts (fuzzy concept lattices) in the sense of traditional logic has been initiated as a generalization of the theory developed by Wille's group [7, 17, 19]. The point of generalization is the non-sharpness (fuzziness) of concepts which is not taken into account in Wille's theory.

Recall that the main purpose of fuzzy logic and fuzzy set theory [13] is to model non-sharp (fuzzy) phenomena. Formally, instead of the two-element Boolean algebra, a more general structure of truth values is employed. In [8, 9], the author proposed to use a structure of a complete residuated lattice which (as turned out later on) plays an important role in fuzzy logic in narrow sense [12, 10, 14].

DEFINITION 2.1
A *complete residuated lattice* is an algebra $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ such that

$\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice with the least element 0 and the greatest element 1,

$\langle L, \otimes, 1\rangle$ is a commutative monoid, i.e. $\otimes$ is commutative, associative, and $x \otimes 1 = x$ holds holds for each $x \in L$, and

$\otimes, \rightarrow$ form an adjoint pair, i.e.

$$x \otimes y \leq z \text{ iff } x \leq y \rightarrow z \tag{2.1}$$

holds for all $x, y, z \in L$.

Residuated lattices have been introduced in [16]. In each residuated lattice it holds that $x \leq y$ implies $x \otimes z \leq y \otimes z$, and $x \leq y$ implies $z \rightarrow x \leq z \rightarrow y$ (isotonicity) and $x \rightarrow z \geq y \rightarrow z$ (antitonicity in the first argument). The operation $\otimes$ is thus a *t-norm* [10, 13], $\rightarrow$ is called *residuum*. In the following we will deal with *complete residuated lattices*, i.e. $\langle L, \wedge, \vee, 0, 1\rangle$ is assumed to be a complete lattice.

The t-norm $\otimes$ and the residuum $\rightarrow$ are intended for modelling of the conjunction and implication, respectively. The reason for using a monoidal conjunction and residuated implication is discussed in [9, 10]. Briefly, the monoidal properties meet the intuitive properties of many-valued conjunction. Obviously, the residuum $\rightarrow$ is uniquely determined by the adjointness property (2.1) which itself corresponds to deduction rule modus ponens [10]. Supremum ($\bigvee$) and infimum ($\bigwedge$) are intended for modelling of the general and existential quantifier, respectively. A semantically complete first-order many-valued logic with semantics defined over complete residuated lattices can be found in [12]. Several special classes of residuated lattices serve as special structures of semantically complete logical calculi (for details see [10]).

The most studied and applied set of truth values is the real interval $[0, 1]$ with $a \wedge b = \min(a, b)$, $a \vee b = \max(a, b)$, and with three important pairs of adjoint operations: the Łukasiewicz one ($a \otimes b = \max(a + b - 1, 0)$, $a \to b = \min(1 - a + b, 1)$), Gödel one ($a \otimes b = \min(a, b)$, $a \to b = 1$ if $a \leq b$ and $= b$ else), and product one ($a \otimes b = a \cdot b$, $a \to b = 1$ if $a \leq b$ and $= b/a$ else). For the role of these 'building stones' in fuzzy logic see [10]. Another important set of truth values is the set (the ordering determines the complete lattice structure) $\{a_0 = 0, a_1, \ldots, a_n = 1\}$ ($a_0 < \cdots < a_n$). Two t-norms are often considered: $a_k \otimes a_l = a_{\max(k+l-n,0)}$ and the corresponding $\to$ given by $a_k \to a_l = a_{\min(n-k+l,n)}$ (Łukasiewicz) and $a_k \otimes a_l = a_{\min(k,l)}$ and the corresponding $\to$ given by $a_k \to a_l = 1$ if $k \leq l$ and a_l else (Gödel). A special case of the latter algebras is the Boolean algebra $\mathbf{2}$ of classical logic with the support $2 = \{0, 1\}$. It may be easily verified that the only t-norm on $\{0, 1\}$ is the classical conjunction operation $\wedge$, i.e. $a \wedge b = 1$ iff $a = 1$ and $b = 1$, which implies that the only residuum operation is the classical implication operation $\to$, i.e. $a \to b = 0$ iff $a = 1$ and $b = 0$. Note that each of the preceding residuated lattices is complete.

The following identities of complete residuated lattices will be needed:

$$a = 1 \to a \tag{2.2}$$

$$a \leq (a \to b) \to b \tag{2.3}$$

$$a \otimes (a \to b) \leq b \tag{2.4}$$

$$(a \otimes b) \to c = a \to (b \to c) = b \to (a \to c) \tag{2.5}$$

$$a \otimes \Big(\bigvee_{j \in J} b_j\Big) = \bigvee_{j \in J} (a \otimes b_j) \tag{2.6}$$

$$a \otimes \bigwedge_{j \in J} b_j \leq \bigwedge_{j \in J} (a \otimes b_j) \tag{2.7}$$

$$\Big(\bigvee_{j \in J} a_j\Big) \to b = \bigwedge_{j \in J} (a_j \to b) \tag{2.8}$$

$$a \to \bigwedge_{j \in J} b_j = \bigwedge_{j \in J} (a \to b_j)\,. \tag{2.9}$$

A *fuzzy set* (or $\mathbf{L}$-set) A in a universe set X is a mapping $A : X \to L$. The value $A(x) \in L$ is interpreted as the truth value of 'the element x belongs to A'. The set of all fuzzy sets in X is denoted by L^X. For $A_1, A_2 \in L^X$ we write $A_1 \subseteq A_2$ iff $A_1(x) \leq A_2(x)$ for all $x \in X$. Similarly, a binary fuzzy relation R between X and Y is a mapping $R : X \times Y \to L$.

We now recall the basic notions of fuzzy concept lattices. The primary notion is that of a *fuzzy context* ($\mathbf{L}$-context): it is a triple $\langle G, M, I\rangle$, where G and M are sets interpreted as the set of objects (G) and the set of attributes (M) to which we restrict our attention, and $I \in L^{G \times M}$ is a fuzzy relation between G and M. The value $I(g, m) \in L$ is interpreted as the truth value of the fact 'the object $g \in G$ has the attribute $m \in M$'. In accordance with the Port-Royal definition, a *(formal) fuzzy concept* ($\mathbf{L}$-concept) is a pair $\langle A, B\rangle$, $A \in L^G$, $B \in L^M$, A plays the role of the extent (fuzzy set of objects which determine the concept), B plays the role of the intent (fuzzy set of attributes which determine the concept), such that:

(a) B is the collection of all attributes shared by all the objects of A

and

(b) A is the collection of all objects having all the attributes of B.

There are therefore two fundamental operators: $^{\uparrow}$ which assigns to each fuzzy set $A \in L^G$ of objects the fuzzy set $A^{\uparrow} \in L^M$ of all the attributes which are common to all objects of A, and $^{\downarrow}$ which assigns to each fuzzy set $B \in L^M$ of attributes the fuzzy set $B^{\downarrow} \in L^G$ of all the objects which which share all the attributes of B). Rewriting these linguistic descriptions on the semantical level of fuzzy logic we get formal definitions

$$A^{\uparrow}(m) = \bigwedge_{g \in G} (A(g) \to I(g,m)) \tag{2.10}$$

and

$$B^{\downarrow}(g) = \bigwedge_{m \in M} (B(m) \to I(g,m)) \tag{2.11}$$

which play the fundamental role. To emphasize the dependence on I, the mappings $^{\uparrow}$ and $^{\downarrow}$ will also be denoted by $^{\uparrow_I}$ and $^{\downarrow_I}$. A (formal) fuzzy concept is therefore a pair $\langle A, B\rangle \in L^G \times L^M$ (of extent A and intent B) such that $A^{\uparrow} = B$ and $B^{\downarrow} = A$. Denote $\mathcal{B}(G, M, I) = \{\langle A, B\rangle \in L^G \times L^M \mid A^{\uparrow} = B, B^{\downarrow} = A\}$ the set of all fuzzy concepts given by the fuzzy context $\langle G, M, I\rangle$.

The conceptual hierarchy discussed above is modelled by the relation $\leq$ defined on $\mathcal{B}(G, M, I)$ by

$$\langle A_1, B_1\rangle \leq \langle A_2, B_2\rangle \text{ iff } A_1 \subseteq A_2 (\text{iff } B_1 \supseteq B_2). \tag{2.12}$$

The set $\mathcal{B}(G, M, I)$ equipped by $\leq$ is called a *fuzzy concept lattice* (**L**-concept lattice). The following theorem describes the hierarchy and characterizes fuzzy concept lattices.

THEOREM 2.2 ([4])
(1) The set $\mathcal{B}(G, M, I)$ is under $\leq$ a complete lattice where the suprema and infima are given by

$$\bigwedge_{j \in J} \langle A_j, B_j\rangle = \langle \bigcap_{j \in J} A_j, (\bigcup_{j \in J} B_j)^{\downarrow\uparrow}\rangle, \tag{2.13}$$

$$\bigvee_{j \in J} \langle A_j, B_j\rangle = \langle (\bigcup_{j \in J} A_j)^{\uparrow\downarrow}, \bigcap_{j \in J} B_j\rangle. \tag{2.14}$$

(2) Moreover, an arbitrary complete lattice $\mathbf{V} = \langle V, \wedge, \vee\rangle$ is isomorphic to some $\mathcal{B}(G, M, I)$ iff there are mappings $\gamma : G \times L \to V$, $\mu : M \times L \to V$ such that

$\gamma(G, L)$ is $\bigvee$-dense in V, $\mu(M, L)$ is $\bigwedge$-dense in V;
$\alpha \otimes \beta \leq I(g, m)$ iff $\gamma(g, \alpha) \leq \mu(m, \beta)$.

Note that $V' \subseteq V$ is $\bigvee$-dense ($\bigwedge$-dense) in V if each $v \in V$ is the supremum (infimum) of some subset of V'. Notice that the lattice structure of concepts is very natural: to each set of concepts there is a concept which is their direct generalization (supremum) and a concept which is their direct specialization (infimum). In this perspective, the original (crisp) concepts lattices [17] are exactly **L**-concept lattices where $L = \{0, 1\}$ is the set of truth values of classical logic. Given a context (i.e. data matrix of logical data obtained from experts), the concept lattice reveals the conceptual structure hidden in the context and provides us with formalism for representing conceptual knowledge [17, 19, 7, 4]. If the information in the context is fuzzy, the fuzzy conceptual data analysis is an appropriate tool for handling it. Fuzzy concept lattice represents a conceptual granulation both of the set of objects and attributes equipped by a hierarchical order.

3 Concepts and similarity

3.1 Similarity relations

A crucial role in the way humans regard the world is played by the *similarity phenomenon*. In fuzzy set theory, similarity phenomenon is approached via so called similarity relations. By a $\otimes$-*similarity relation* (or fuzzy $\otimes$-equivalence relation, L-valued global equality) [13, 12, 20] on a universe U it is meant a binary fuzzy relation E satisfying the following properties for all $x, y, z \in U$:

$$E(x,x) = 1 \tag{3.1}$$
$$E(x,y) = E(y,x) \tag{3.2}$$
$$E(x,y) \otimes E(y,z) \leq E(x,z). \tag{3.3}$$

Properties (3.1), (3.2) and (3.3) are called reflexivity, symmetry, and transitivity, respectively. The $\otimes$-*similarity class* of $x \in U$ is the fuzzy set $[x]_E \in L^U$ given by $[x]_E(y) = E(x,y)$ for each $y \in U$, i.e. it is a collection of elements similar to x. A fuzzy set $A \in L^G$ is said to be *extensional* w.r.t. E if for every $x, y \in U$ it holds $A(x) \otimes E(x,y) \leq A(y)$, i.e. if with each element x, A contains all the elements similar to x. In this case, E is also said to be *compatible* with A. Non-extensional fuzzy sets are not compatible with the underlying similarity relation. It is easily seen that in the crisp case, i.e. $L = \{0,1\}$, similarity relations are equivalence relations. For the study of similarity phenomenon, the crisp case is a degenerate one and non-interesting—two elements x and y may be 'fully similar' ($E(x,y) = 1$) or 'fully dissimilar' ($E(x,y) = 0$). Note that in the crisp case, tolerance relations [5] (i.e. reflexive and symmetric relations) are used instead of equivalence relations for modelling non-transitive similarities (e.g. two words are held to be similar if they differ by at most one letter). Tolerance relations may be considered also from the point of view of fuzzy approach, i.e. requiring only (3.1) and (3.2). We will not follow this way since all of the subsequently studied relations are defined by the generic rule (3.4) of Lemma 3.1 and are therefore transitive.

To be able to model the equivalence of truth values we have at our disposal the so called *biresiduum* (or biimplication) [12, 14] operation $\leftrightarrow$ defined by

$$a \leftrightarrow b = (a \to b) \wedge (b \to a).$$

The following lemma will be useful in the following considerations.

LEMMA 3.1
Let E be a $\otimes$-similarity on U, $\mathcal{S} = \{A_i \in L^U \mid i \in I\}$ be a family of fuzzy sets. (1) E is the largest $\otimes$-similarity relation compatible with all $[x]_E$. (2) The relation $E_{\mathcal{S}}$ defined by

$$E_{\mathcal{S}}(x,y) = \bigwedge_{i \in I} (A_i(x) \leftrightarrow A_i(y)) \tag{3.4}$$

is the largest $\otimes$-similarity relation compatible with all $A_i \in \mathcal{S}$. Moreover, $A_i(x) = 1$ implies $[x]_{E_\mathcal{S}} \subseteq A_i$.

PROOF. (1) is an immediate consequence of (2) provided $E = E_{\{[x]_E \mid x \in U\}}$. Denote $E' = E_{\{[x]_E \mid x \in U\}}$. We have $E(y,z) \le E'(y,z) = \bigwedge_{x \in U}(E(x,y) \leftrightarrow E(x,z))$ iff for each $x \in U$ it holds $E(y,z) \le E(x,y) \leftrightarrow E(x,z) = (E(x,y) \to E(x,z)) \wedge (E(x,z) \to E(x,y))$ iff both $E(x,y) \le E(x,y) \to E(x,z)$ and $E(x,y) \le E(x,z) \to E(x,y)$ which is evident by applying the adjointness and transitivity. On the other hand, taking $x = y$ we get $E'(y,z) \le E(y,y) \leftrightarrow E(y,z) = E(y,z)$, i.e. $E = E'$ proving (1).

(2), We have to check that $E_\mathcal{S}$ satisfies (3.1)–(3.3). (3.1) follows from $a \leftrightarrow a = 1$ and (3.2) follows from $a \leftrightarrow b = b \leftrightarrow a$. Furthermore,

$$
\begin{aligned}
& E_\mathcal{S}(x,y) \otimes E_\mathcal{S}(y,z) \\
= {} & \Big(\bigwedge_{i \in I}(A_i(x) \leftrightarrow A_i(y))\Big) \otimes \Big(\bigwedge_{j \in I}(A_j(y) \leftrightarrow A_j(z))\Big) && (3.5) \\
\le {} & \bigwedge_{i \in I}\Big(\Big(A_i(x) \leftrightarrow A_i(y)\Big) \otimes \bigwedge_{j \in I}\Big(A_j(y) \leftrightarrow A_j(z)\Big)\Big) && (3.6) \\
\le {} & \bigwedge_{i \in I}\bigwedge_{j \in I}\Big(\Big(A_i(x) \leftrightarrow A_i(y)\Big) \otimes \Big(A_j(y) \leftrightarrow A_j(z)\Big)\Big) && (3.7) \\
\le {} & \bigwedge_{i \in I}\Big(\Big(A_i(x) \leftrightarrow A_i(y)\Big) \otimes \Big(A_i(y) \leftrightarrow A_i(z)\Big) \\
= {} & \bigwedge_{i \in I}\Big(\Big\{\Big(A_i(x) \to A_i(y)\Big) \wedge \Big(A_i(y) \to A_i(x)\Big)\Big\} && (3.8) \\
& \otimes\Big\{\Big(A_i(y) \to A_i(z)\Big) \wedge \Big(A_i(z) \to A_i(y)\Big)\Big\}\Big) \\
\le {} & \bigwedge_{i \in I}\Big(\Big\{\Big(A_i(x) \to A_i(y)\Big) \otimes \Big(A_i(y) \to A_i(z)\Big)\Big\} && (3.9) \\
& \wedge\Big\{\Big(A_i(x) \to A_i(y)\Big) \otimes \Big(A_i(z) \to A_i(y)\Big)\Big\} \\
& \wedge\Big\{\Big(A_i(y) \to A_i(x)\Big) \otimes \Big(A_i(y) \to A_i(z)\Big)\Big\} \\
& \wedge\Big\{\Big(A_i(y) \to A_i(x)\Big) \otimes \Big(A_i(z) \to A_i(y)\Big)\Big\}\Big) \\
\le {} & \bigwedge_{i \in I}\Big(\Big\{\Big(A_i(x) \to A_i(y)\Big) \otimes \Big(A_i(y) \to A_i(z)\Big)\Big\} && (3.10) \\
& \wedge\Big\{\Big(A_i(z) \to A_i(y)\Big) \otimes \Big(A_i(y) \to A_i(x)\Big)\Big\}\Big) \\
\le {} & \bigwedge_{i \in I}\Big(\Big(A_i(x) \to A_i(z)\Big) \wedge \Big(A_i(z) \to A_i(x)\Big)\Big) && (3.11) \\
= {} & \bigwedge_{i \in I}\Big(A_i(x) \leftrightarrow A_i(z)\Big) = E_\mathcal{S}(x,z)\,,
\end{aligned}
$$

hence (3.3) holds. In (3.5)–(3.6), (3.6)–(3.7) and (3.8)–(3.9) we used the fact $a \otimes \bigwedge_{i \in I} b_i \le$

$\bigwedge_{i\in I}(a \otimes b_i)$. In (3.10)–(3.11) we used $(a \to b) \otimes (b \to c) \le a \to c$. The extensionality of each $A_i \in \mathcal{S}$ follows by $A_i(x) \otimes E_{\mathcal{S}} \le A_i(x) \otimes (A_i(x) \to A_i(y)) \le A_i(y)$. If all A_i are extensional to another $\otimes$-similarity E, then $A_i(x) \otimes E(x,y) \le A_i(y)$, hence $E(x,y) \le A_i(x) \to A_i(y)$, and, similarly, $E(x,y) \le A_i(y) \to A_i(x)$, i.e. $E(x,y) \le A_i(x) \leftrightarrow A_i(y)$ hold for all $x, y \in U$. We conclude $E(x,y) \le \bigwedge_{i\in I}(A_i(x) \leftrightarrow A_i(y)) = E_{\mathcal{S}}(x,y)$, i.e. $E_{\mathcal{S}}$ is the largest one. Finally, if $A_i(x) = 1$ then $[x]_{E_{\mathcal{S}}}(y) = E_{\mathcal{S}}(x,y) \le A_i(x) \leftrightarrow A_i(y) = 1 \leftrightarrow A_i(y) = A_i(y)$, i.e. $[x]_{E_{\mathcal{S}}} \subseteq A_i$. The proof is finished. ∎

Notice that for the crisp case (i.e. $L = \{0,1\}$), $E_{\mathcal{S}}$ is a crisp equivalence relation—two elements of the universe are equivalent iff there is no set of the family which separates them.

3.2 Similarity of objects and attributes

We are going to propose a way to measure similarity of objects and similarity of attributes of a given **L**-context. This similarity will be induced by the structure of **L**-concepts determined by the **L**-context. We prove that the similarity may be determined directly from the **L**-context which is relevant from the computational point of view.

We are given (in some sense relevant) objects (elements of G) and their (observed) attributes (elements of M). The given **L**-context gives rise to a complete lattice of all induced **L**-concepts. The idea of conceptual classification leads to the use of the induced conceptual structure $\mathcal{B}(G,M,I)$ to define similarity relations on G and on M.

Consider the problem of similarity of objects. Informally, two objects $g_1, g_2 \in G$ are similar if they cannot be separated by any concept, i.e. if for each concept c it holds that g_1 belongs to the extent of c iff g_2 belongs to the extent of c. This leads to the following definition of a relation $E^G_{\mathcal{B}(G,M,I)} \in L^{G\times G}$:

$$E^G_{\mathcal{B}(G,M,I)}(g_1,g_2) = \bigwedge_{\langle A,B\rangle\in\mathcal{B}(G,M,I)} \Big(A(g_1) \leftrightarrow A(g_2)\Big)\,. \tag{3.12}$$

The relation $E^G_{\mathcal{B}(G,M,I)}$ will be called *induced* (by $\mathcal{B}(G,M,I)$) *similarity on* G. By Lemma 3.1 we immediately get the following statement.

THEOREM 3.2
The relation $E^G_{\mathcal{B}(G,M,I)}$ is the largest $\otimes$-similarity relation on G compatible with the extents of all concepts of $\mathcal{B}(G,M,I)$.

From the computational point of view (we always assume both G and M to be finite if algorithmic aspects are concerned), the foregoing definition leads to the following algorithm for computing the similarity relation $E^G_{\mathcal{B}(G,M,I)}$: Take an **L**-context, generate all the **L**-concepts of $\mathcal{B}(G,M,I)$ and determine the similarity of each pair $\langle g_1, g_2\rangle \in G\times G$ by (3.12). The **L**-concept lattice may be, however, quite extensive. This poses the question whether the computational cost can be reduced. We give a (exact) solution which reduces the computational costs significantly. Define a relation $E^G_{\langle G,M,I\rangle} \in L^{G\times G}$ by

$$E^G_{\langle G,M,I\rangle}(g_1,g_2) = \bigwedge_{m\in M} \left(I(g_1,m) \leftrightarrow I(g_2,m)\right). \tag{3.13}$$

$E^G_{\langle G,M,I\rangle}(g_1,g_2)$ may be obtained from the **L**-context $\langle G,M,I\rangle$ computing $|M|$ times the operation $\leftrightarrow$. Using Lemma 3.1 (put $X = G$, $I = M$, $A_i(g) = I(g,m)$) we get

THEOREM 3.3
The relation $E^G_{\langle G,M,I\rangle}$ is the largest $\otimes$-similarity relation on G compatible with all $I(\_,m) \in L^G$, $m \in M$.

The following theorem solves the problem of finding an efficient procedure for computing the similarity relation $E^G_{\mathcal{B}(G,M,I)}$.

THEOREM 3.4
Let $\langle G, M, I\rangle$ be a **L**-context. Then for the similarity relations defined by (3.13) and (3.12) it holds

$$E^G_{\mathcal{B}(G,M,I)} = E^G_{\langle G,M,I\rangle} \,. \tag{3.14}$$

PROOF. We show $\bigwedge_{m\in M}(I(g_1,m) \leftrightarrow I(g_2,m)) = \bigwedge_{\langle A,B\rangle\in\mathcal{B}(G,M,I)}\Big(A(g_1) \leftrightarrow A(g_2)\Big)$ by checking both inequalities.

Consider the concept $\langle A_m, B_m\rangle = \langle\{\,{}^1/m\}^{\downarrow}, \{\,{}^1/m\}^{\downarrow\uparrow}\rangle$ for $m \in M$. For each $g \in G$ it holds

$$\begin{aligned} A_m(g) &= \bigwedge\{\,\{\,{}^1/m\}(m') \to I(g,m);\, m' \in M\} \\ &= 1 \to I(g,m) = I(g,m)\,. \end{aligned}$$

From $\{\langle A_m, B_m\rangle \mid m \in M\} \subseteq \mathcal{B}(G,M,I)$ and from the properties of infimum it follows that

$$\begin{aligned} &\bigwedge_{\langle A,B\rangle\in\mathcal{B}(G,M,I)} \Big(A(g_1) \leftrightarrow A(g_2)\Big) \le \bigwedge_{m\in M} \Big(A_m(g_1) \leftrightarrow A_m(g_2)\Big) \\ =\ &\bigwedge_{m\in M} (I(g_1,m) \leftrightarrow I(g_2,m)) \end{aligned}$$

for all $g_1, g_2 \in G$, proving the first inequality.

Conversely,

$$\bigwedge_{m\in M} (I(g_1,m) \leftrightarrow I(g_2,m)) \le \bigwedge_{\langle A,B\rangle\in\mathcal{B}(G,M,I)} \Big(A(g_1) \leftrightarrow A(g_2)\Big)$$

iff for each $\langle A, B\rangle \in \mathcal{B}(G,M,I)$ it holds

$$\begin{aligned} &\bigwedge_{m\in M} (I(g_1,m) \leftrightarrow I(g_2,m)) \\ \le\ & A(g_1) \leftrightarrow A(g_2) = (A(g_1) \to A(g_2)) \wedge ((A(g_2) \to A(g_1)) \end{aligned}$$

which holds iff both

$$\bigwedge_{m\in M} (I(g_1,m) \leftrightarrow I(g_2,m)) \le A(g_1) \to A(g_2) \tag{3.15}$$

and

$$\bigwedge_{m\in M} (I(g_1,m) \leftrightarrow I(g_2,m)) \le A(g_2) \to A(g_1) \tag{3.16}$$

hold. (3.15) holds iff

$$A(g_1) \otimes \bigwedge_{m \in M} (I(g_1, m) \leftrightarrow I(g_2, m)) \leq A(g_2).$$

Since $\langle A, B\rangle \in \mathcal{B}(G, M, I)$, i.e. $A = A^{\uparrow\downarrow}$, the last inequality means

$$\begin{aligned} &\bigwedge_{m \in M} (A^{\uparrow}(m) \to I(g_1, m)) \otimes \bigwedge_{m \in M} (I(g_1, m) \leftrightarrow I(g_2, m)) \\ \leq \quad &\bigwedge_{m \in M} (A^{\uparrow}(m) \to I(g_2, m)) \end{aligned}$$

which holds iff for each $m' \in M$ we have

$$\begin{aligned} &\bigwedge_{m \in M} (A^{\uparrow}(m) \to I(g_1, m)) \otimes \bigwedge_{m \in M} (I(g_1, m) \leftrightarrow I(g_2, m)) \\ \leq \quad &A^{\uparrow}(m') \to I(g_2, m'), \end{aligned}$$

i.e.

$$\begin{aligned} &A^{\uparrow}(m') \otimes \bigwedge_{m \in M} (A^{\uparrow}(m) \to I(g_1, m)) \otimes \bigwedge_{m \in M} (I(g_1, m) \leftrightarrow I(g_2, m)) \\ \leq \quad &I(g_2, m). \end{aligned} \tag{3.17}$$

Evidently, by (2.4),

$$\begin{aligned} &A^{\uparrow}(m') \otimes \bigwedge_{m \in M} (A^{\uparrow}(m) \to I(g_1, m)) \otimes \bigwedge_{m \in M} (I(g_1, m) \leftrightarrow I(g_2, m)) \\ &\leq A^{\uparrow}(m') \otimes (A^{\uparrow}(m') \to I(g_1, m')) \otimes (I(g_1, m') \to I(g_2, m')) \leq I(g_2, m'), \end{aligned}$$

thus (3.17) and therefore also (3.15) hold. (3.16) may be proved analogously. The second inequality and hence also the theorem is proved. ■

From the foregoing theorem we have also the following consequence which is in accordance with our intuition: If we are given family of (elementary) properties (attributes) of objects and consider the structure of concepts which is given by these properties then the similarity among objects considered w.r.t. the structure of concepts is the same as the similarity w.r.t. to only the basic properties.

In a completely analogous way we may get the following results for the similarity relations on M.

THEOREM 3.5
For an **L**-context $\langle G, M, I\rangle$, the relations $E^M_{\mathcal{B}(G,M,I)}$ and $E^M_{\langle G,M,I\rangle}$ defined by

$$E^M_{\mathcal{B}(G,M,I)}(m_1, m_2) = \bigwedge_{\langle A,B\rangle \in \mathcal{B}(G,M,I)} (B(m_1) \leftrightarrow B(m_2))$$

and

$$E^M_{\langle G,M,I\rangle}(m_1, m_2) = \bigwedge_{g \in G} (I(g, m_1) \leftrightarrow I(g, m_2))$$

are $\otimes$-similarity relations on M and it holds $E^M_{\mathcal{B}(G,M,I)} = E^M_{\langle G,M,I\rangle}$. They are the largest $\otimes$-similarity relations compatible with the intents of all concepts of $\mathcal{B}(G, M, I)$.

3.3 Similarity of concepts

The next level on which the similarity phenomenon will be considered is the level of concepts. Observe first the following fact.

LEMMA 3.6
For any universe U, the relation E on L^U given for any $A_1, A_2 \in L^U$ by

$$E(A_1, A_2) = \bigwedge_{x \in U} (A_1(x) \leftrightarrow A_2(x))$$

is the largest $\otimes$-similarity relation on L^U such that $A_1(x) \otimes E(A_1, A_2) \leq A_2(x)$ holds for each $x \in U$, $A_1, A_2 \in L^U$.

PROOF. Putting $I = U$, $X = L^U$, $A_i(x) = x(i)$ for $x \in L^U$, $i \in U$, the assertion is a direct consequence of Lemma 3.1. ∎

In the following, it will be clear what universe U the relation E concerns.

Consider first the relations E^{Ext} and E^{Int} on $\mathcal{B}(G, M, I)$, call them *induced similarity* by extents and *induced similarity* by intents, respectively:

$$E^{Ext}(\langle A_1, B_1\rangle, \langle A_2, B_2\rangle) = E(A_1, A_2) = \bigwedge_{g \in G} (A_1(g) \leftrightarrow A_2(g)),$$

$$E^{Int}(\langle A_1, B_1\rangle, \langle A_2, B_2\rangle) = E(B_1, B_2) = \bigwedge_{m \in M} (B_1(m) \leftrightarrow B_2(m)).$$

Lemma 3.6 gives immediately the following statement.

THEOREM 3.7
E^{Ext} and E^{Int} are the largest $\otimes$-similarity relations on $\mathcal{B}(G, M, I)$ such that $A_1(g) \otimes E^{Ext}(\langle A_1, B_1\rangle, \langle A_2, B_2\rangle) \leq A_2(g)$ and $B_1(m) \otimes E^{Int}(\langle A_1, B_1\rangle, \langle A_2, B_2\rangle) \leq B_2(m)$ hold for every $g \in G$, $m \in M$, $\langle A_1, B_1\rangle$, $\langle A_2, B_2\rangle \in \mathcal{B}(G, M, I)$.

To answer the question of how the relations E^{Ext} and E^{Int} are related, we derive some preliminary results. The next lemma states that the operators $^{\uparrow}$ and $^{\downarrow}$ preserve similarity.

LEMMA 3.8
Let $\langle G, M, I\rangle$ be an $\mathbf{L}$-context, $A_1, A_2 \in L^G$, $B_1, B_2 \in L^M$. Then it holds $E(A_1, A_2) \leq E(A_1^{\uparrow}, A_2^{\uparrow})$ and $E(B_1, B_2) \leq E(B_1^{\downarrow}, B_2^{\downarrow})$.

PROOF. We prove only $E(A_1, A_2) \leq E(A_1^{\uparrow}, A_2^{\uparrow})$, the second part may be obtained symmetrically.

$$\begin{aligned} \bigwedge_{g \in G} (A_1(g) \leftrightarrow A_2(g)) &= E(A_1, A_2) \\ \leq \quad E(A_1^{\uparrow}, A_2^{\uparrow}) &= \bigwedge_{m \in M} \left(A_1^{\uparrow}(m) \leftrightarrow A_2^{\uparrow}(m)\right) \end{aligned}$$

holds iff for each $m \in M$ it holds

$$\begin{aligned} &\bigwedge_{g \in G} (A_1(g) \leftrightarrow A_2(g)) \\ &\leq \quad A_1^{\uparrow}(m) \leftrightarrow A_2^{\uparrow}(m) = \left(A_1^{\uparrow}(m) \to A_2^{\uparrow}(m)\right) \wedge \left(A_2^{\uparrow}(m) \to A_1^{\uparrow}(m)\right), \end{aligned}$$

which holds iff the left side of the inequality is less than or equal to both members of the right side which are connected by $\wedge$. We check only the first of these inequalities, i.e.

$$\bigwedge_{g\in G}(A_1(g)\leftrightarrow A_2(g))\leq A_1^{\uparrow}(m)\rightarrow A_2^{\uparrow}(m).$$

By adjunction, this holds iff

$$A_1^{\uparrow}(m)\otimes\bigwedge_{g\in G}(A_1(g)\leftrightarrow A_2(g))\leq A_2^{\uparrow}(m),$$

i.e.

$$\left(\bigwedge_{g\in G}A_1(g)\rightarrow I(g,m)\right)\otimes\bigwedge_{g\in G}(A_1(g)\leftrightarrow A_2(g))\leq\bigwedge_{g\in G}A_2(g)\rightarrow I(g,m),$$

which holds iff for each $g'\in G$ the inequality

$$\bigwedge_{g\in G}(A_1(g)\leftrightarrow A_2(g))\otimes\left(\bigwedge_{g\in G}A_1(g)\rightarrow I(g,m)\right)\leq A_2(g')\rightarrow I(g',m)$$

holds. The last inequality is equivalent (by adjunction) to

$$A_2(g')\otimes\bigwedge_{g\in G}(A_1(g)\leftrightarrow A_2(g))\otimes\left(\bigwedge_{g\in G}A_1(g)\rightarrow I(g,m)\right)\leq I(g',m),$$

which holds because

$$\begin{aligned}&A_2(g')\otimes\bigwedge_{g\in G}(A_1(g)\leftrightarrow A_2(g))\otimes\left(\bigwedge_{g\in G}A_1(g)\rightarrow I(g,m)\right)\\ \leq\quad&A_2(g')\otimes(A_2(g')\rightarrow A_1(g'))\otimes(A_1(g')\rightarrow I(g',m))\leq I(g',m),\end{aligned}$$

by applying twice the rule (2.4). ∎

The following corollary is immediate.

COROLLARY 3.9
Under the conditions of Lemma 3.8, it holds $E(A_1,A_2)\leq E(A_1^{\uparrow\downarrow},A_2^{\uparrow\downarrow})$ and $E(B_1,B_2)\leq E(B_1^{\downarrow\uparrow},B_2^{\downarrow\uparrow})$.

Note that even without Lemma 3.8 we have, due to the properties of E, that $E(A_1^{\uparrow\downarrow},A_1)\otimes E(A_1,A_2)\otimes E(A_2,A_2^{\uparrow\downarrow})\leq E(A_1^{\uparrow\downarrow},A_2^{\uparrow\downarrow})$, i.e. if $A_1^{\uparrow\downarrow}$ and A_1, A_1 and A_2, A_2 and $A_2^{\uparrow\downarrow}$ are pairwise similar then also $A_1^{\uparrow\downarrow}$ and $A_2^{\uparrow\downarrow}$ are similar. Corollary 3.9 asserts a stronger result.

COROLLARY 3.10
Under the conditions of Lemma 3.8, it holds $E(A_1,A_2)\otimes E(A_1,A_1^{\uparrow\downarrow})\leq E(A_1,A_2^{\uparrow\downarrow})$ and $E(B_1,B_2)\otimes E(B_1,B_1^{\downarrow\uparrow})\leq E(B_1,B_2^{\downarrow\uparrow})$.

PROOF. $E(A_1,A_2)\otimes E(A_1,A_1^{\uparrow\downarrow}) \le E(A_1^{\uparrow\downarrow},A_2^{\uparrow\downarrow})\otimes E(A_1,A_1^{\uparrow\downarrow}) \le E(A_1,A_2^{\uparrow\downarrow})$. The second part may be proved analogously. ■

The following result shows that the similarities of concepts by extents and intents are equal.

THEOREM 3.11
For any **L**-context $\langle G,M,I\rangle$ it holds $E^{Ext} = E^{Int}$.

PROOF. Let $\langle A_1,B_1\rangle,\langle A_1,B_1\rangle \in \mathcal{B}(G,M,I)$, i.e. $A_i^{\uparrow} = B_i$ and $B_i^{\downarrow} = A_i$ for $i = 1,2$. By Lemma 3.8 we get

$$\begin{aligned} E^{Ext}(\langle A_1,B_1\rangle,\langle A_2,B_2\rangle) &= E(A_1,A_2) \\ \le\quad E(A_1^{\uparrow},A_2^{\uparrow}) &= E(B_1,B_2) = E^{Int}(\langle A_1,B_1\rangle,\langle A_2,B_2\rangle), \end{aligned}$$

and analogously, $E^{Int}(\langle A_1,B_1\rangle,\langle A_2,B_2\rangle) \le E^{Ext}(\langle A_1,B_1\rangle,\langle A_2,B_2\rangle)$.
To sum up, $E^{Ext}(\langle A_1,B_1\rangle,\langle A_2,B_2\rangle) = E^{Int}(\langle A_1,B_1\rangle,\langle A_2,B_2\rangle)$. ■

We will therefore write E instead of E^{Ext} and E^{Int} and call it the *induced similarity* on concepts.

Compatible similarities and factorization

The primary importance of similarity relations in human reasoning is the *reduction of the complexity of the outer world at a reasonable price*. The complexity is reduced via considering the 'collections of similar elements in concern' rather than the particular elements themselves. This is just what is in the general system theory known as the *abstraction process by factorization*: moving from a given level of abstraction (distinguishability) one level up where the elements are collections of elements of the lower level. Instead of the original system one therefore considers the 'system modulo similarity'. The price paid is the loss of precision.

Our concern in the following is the *reduction of the complexity of the concept lattice by factorization modulo similarity*. The concept lattice of a given context represents the overall conceptual structure which can be considerably intricate. To get an insight one has to look for methods for reducing the complexity of the structure. In the two-valued (sharp) case, a considerable attention has been paid to this problem [7, 17]. In the many-valued (fuzzy) case, one would expect methods for gradual reduction of the complexity. The idea is to factorize the concept lattice by appropriate a-cut ${}^aE$ of the similarity E (note that ${}^aE = \{\langle c_1,c_2\rangle \mid a \le E(c_1,c_2)\}$ [13]), controlling thus the complexity by $a \in L$. Clearly, the lower $a \in L$, the coarser the factorization. The process of factorization of a system consists of two steps. First, specification of the elements, and, secondly, specification of the structure of the factor system. Since both of the steps are non-standard in our case we will describe them in more detail. In general, algebraic systems can be factorized be *congruences*, i.e. equivalences compatible with the structure of the system. We deal with conceptual structures which are complete lattices. The a-cut ${}^aE$ is clearly a tolerance relation (i.e. reflexive and symmetric), not transitive in general. Compatible tolerance relations on universal algebras have been extensively studied by Chajda [5]. In general, factorization of algebras by compatible tolerances is not possible. Surprisingly, Czédli [6] showed a way to factorize lattices by compatible tolerance relations. The construction has been then used for the factorization of complete lattices (and hence sharp concept lattices) [18]. In the following we describe the construction of the factor lattice of an **L**-concept lattice by a compatible tolerance relation. Let $\langle G,M,I\rangle$ be an **L**-context. A tolerance relation T on $\mathcal{B}(G,M,I)$ is said to be *compatible* if it is preserved under arbitrary suprema and infima, i.e. if $\langle c_j,c'_j\rangle\in T$, $j\in J$, implies both $\langle\bigvee_{j\in J}c_j,\bigvee_{j\in J}c'_j\rangle\in T$ and $\langle\bigwedge_{j\in J}c_j,\bigwedge_{j\in J}c'_j\rangle\in T$ for any $c_j,c'_j\in\mathcal{B}(G,M,I)$, $j\in J$. For a compatible tolerance relation T on $\mathcal{B}(G,M,I)$ denote $c_T=\bigwedge_{\langle c,c'\rangle\in T}c'$ and $c^T=\bigvee_{\langle c,c'\rangle\in T}c'$. Call $[c]_T=[c_T,(c_T)^T]=\{c'\in\mathcal{B}(G,M,I)\mid c_T\leq c'\leq(c_T)^T\}$ a *block* of T and denote $\mathcal{B}(G,M,I)/T=\{[c]_T\mid c\in\mathcal{B}(G,M,I)\}$ the set of all blocks. Introduce a relation $\leq_T$ on $\mathcal{B}(G,M,I)/T$ by $[c]_T\leq_T[c']_T$ iff $\bigwedge[c]_T\leq\bigwedge[c']_T$ (iff $\bigvee[c]_T\leq\bigvee[c']_T$). The justification of the construction is given by the following statement which follows immediately from [18].

PROPOSITION 3.12
(1) $\mathcal{B}(G,M,I)/T$ is the set of all maximal tolerance blocks, i.e. $\mathcal{B}(G,M,I)/T=\{B\subseteq\mathcal{B}(G,M,I)\mid(B\times B\subseteq T)\&((\forall B'\supset B)(B'\times B'\not\subseteq T))\}$. (2) $\langle\mathcal{B}(G,M,I)/T,\leq_T\rangle$ is a complete lattice (factor lattice) where suprema and infima are described by

$$\bigvee_{j\in J}[c_j]_T=[\bigvee_{j\in J}c_j]_T\quad\text{and}\quad\bigwedge_{j\in J}[c_j]_T=[(\bigwedge_{j\in J}c_j)^T]_T\tag{3.18}$$

for every $c_j\in\mathcal{B}(G,M,I)$, $j\in J$.

Substituing (2.13) and (2.14) into (3.18) we get a more concrete description of the lattice operations.

Coming back to the induced similarity E on $\mathcal{B}(G,M,I)$, the ultimate question is that of the compatibility of the a-cuts of E. Call a $\otimes$-similarity relation F on $\mathcal{B}(G,M,I)$ *compatible* if ${}^aE$ is a compatible tolerance relation on $\mathcal{B}(G,M,I)$ for each $a\in L$. Notice that for the two-valued (crisp) case the situation is completely uninteresting. Namely, as one easily checks, the only cases are ${}^0E=\mathcal{B}(G,M,I)\times\mathcal{B}(G,M,I)$ and ${}^1E=\mathrm{id}_{\mathcal{B}(G,M,I)}=\{\langle c,c\rangle\mid c\in\mathcal{B}(G,M,I)\}$. In the first case, $\mathcal{B}(G,M,I)/{}^0E=\{\mathcal{B}(G,M,I)\}$, i.e. the factor lattice collapsed into a one element lattice, while in the second case, $\mathcal{B}(G,M,I)/{}^1E=\{\{\langle A,B\rangle\}\mid\langle A,B\rangle\in\mathcal{B}(G,M,I)\}$, i.e. $\mathcal{B}(G,M,I)$ and $\mathcal{B}(G,M,I)/{}^1E$ are isomorphic.

Note that we have in no case to confine ourselves to the induced similarity E. On the other hand, taking into account only $\otimes$-similarity relations F satisfying $A(g)\otimes F(\langle A,B\rangle,\langle A',B'\rangle)\leq A'(g)$ (which is quite natural—it reads 'object belonging to the extent of some concept belongs also to the extent of any similar concept') for each $g\in G$, Theorem 3.7 tells us that E provides the most extensive reduction: for any other F and each $a\in L$, ${}^aE$ is coarser than ${}^aF$.

We will make use of the following lemma.

LEMMA 3.13
For every $\langle A_j,B_j\rangle,\langle A'_j,B'_j\rangle\in\mathcal{B}(G,M,I)$, $j\in J$, it holds

$$\bigwedge_{j\in J}E(\langle A_j,B_j\rangle,\langle A'_j,B'_j\rangle)\leq E(\bigwedge_{j\in J}\langle A_j,B_j\rangle,\bigwedge_{j\in J}\langle A'_j,B'_j\rangle),\tag{3.19}$$

$$\bigwedge_{j\in J}E(\langle A_j,B_j\rangle,\langle A'_j,B'_j\rangle)\leq E(\bigvee_{j\in J}\langle A_j,B_j\rangle,\bigvee_{j\in J}\langle A'_j,B'_j\rangle).\tag{3.20}$$

PROOF. By the above statements, (3.19) holds iff

$$\bigwedge_{j\in J}E(A_j,A'_j)\leq E(\bigwedge_{j\in J}A_j,\bigwedge_{j\in J}A'_j),$$

i.e.

$$\bigwedge_{j\in J}(\bigwedge_{g\in G} A_j(g) \leftrightarrow A'_j(g)) \leq \bigwedge_{g\in G}(\bigwedge_{j\in J} A_j(g)) \leftrightarrow (\bigwedge_{j\in J} A'_j(g)),$$

which holds iff for each $g' \in G$ we have

$$\begin{aligned}\bigwedge_{j\in J}\bigwedge_{g\in G} A_j(g) \leftrightarrow A'_j(g) &\leq (\bigwedge_{j\in J} A_j(g')) \leftrightarrow (\bigwedge_{j\in J} A'_j(g')) \\ &= (\bigwedge_{j\in J} A_j(g')) \to (\bigwedge_{j\in J} A'_j(g')) \wedge (\bigwedge_{j\in J} A'_j(g')) \to (\bigwedge_{j\in J} A_j(g')).\end{aligned}$$

The last inequality holds iff the left side is less than or equal to both of the conjuncts on the right side. As they can be handled analogously, we prove only

$$\bigwedge_{j\in J}\bigwedge_{g\in G}(A_j(g) \leftrightarrow A'_j(g)) \leq (\bigwedge_{j\in J} A_j(g')) \leftrightarrow (\bigwedge_{j\in J} A'_j(g')),$$

which is, by adjunction, equivalent to

$$(\bigwedge_{j\in J} A_j(g')) \otimes \bigwedge_{j\in J}\bigwedge_{g\in G}(A_j(g) \leftrightarrow A'_j(g)) \leq (\bigwedge_{j\in J} A'_j(g')),$$

which holds since

$$\begin{aligned}&(\bigwedge_{j\in J} A_j(g')) \otimes \bigwedge_{j\in J}\bigwedge_{g\in G}(A_j(g) \leftrightarrow A'_j(g)) \\ \leq\ & (\bigwedge_{j\in J} A_j(g')) \otimes \bigwedge_{j\in J}(A_j(g') \to A'_j(g')) \\ \leq\ & \bigwedge_{j\in J}(\bigwedge_{j'\in J} A'_j(g') \otimes (A_j(g') \to A'_j(g')) \\ \leq\ & \bigwedge_{j\in J}(A_j(g') \otimes (A_j(g') \to A'_j(g')) \leq \bigwedge_{j\in J} A'_j(g').\end{aligned}$$

We have proved (3.19). (3.20) can be proved symmetrically using the fact $\bigvee_{j\in J}\langle A_j, B_j\rangle = \left\langle (\bigcup_{j\in J} A_j)^{\uparrow\downarrow}, \bigwedge_{j\in J} B_j \right\rangle$ of Theorem 2.2 and proceeding by the similarity E on the intents. ■

Lemma 3.13 has an interesting corollary stating that the similarity of any two concepts is less than or equal to the similarity of any of them to their direct join or meet.

COROLLARY 3.14
Let $\langle A_1, B_1\rangle, \langle A_2, B_2\rangle \in \mathcal{B}(G, M, I)$. The following inequalities hold for $i = 1, 2$:

$$\begin{aligned}E(\langle A_1, B_1\rangle, \langle A_2, B_2\rangle) &\leq E(\langle A_i, B_i\rangle, \langle A_1, B_1\rangle \wedge \langle A_2, B_2\rangle), \\ E(\langle A_1, B_1\rangle, \langle A_2, B_2\rangle) &\leq E(\langle A_i, B_i\rangle, \langle A_1, B_1\rangle \vee \langle A_2, B_2\rangle).\end{aligned}$$

PROOF. Put $J = \{x, y\}$, $\langle A_x, B_x\rangle = \langle A_y, B_y\rangle = \langle A_1, B_1\rangle$, $\langle A'_x, B'_x\rangle = \langle A_1, B_1\rangle$, $\langle A'_y, B'_y\rangle = \langle A_2, B_2\rangle$ and apply Lemma 3.13. ■

THEOREM 3.15
The induced similarity E on $\mathcal{B}(G,M,I)$ is compatible. If $a \in L$ is $\otimes$-idempotent (i.e. $a \otimes a = a$) then ${}^{a}E$ is, moreover, transitive, i.e. a congruence relation on $\mathcal{B}(G,M,I)$.

PROOF. The first part follows immediately from (3.19) and (3.20) by the fact that if $a \leq E(\langle A_j, B_j\rangle, \langle A'_j, B'_j\rangle)$ for $j \in J$ then also $a \leq \bigwedge_{j \in J} E(\langle A_j, B_j\rangle, \langle A'_j, B'_j\rangle)$. The second part follows from the evident fact that if a is $\otimes$-idempotent and $a \leq b, c$ then $a \leq b \otimes c$. ∎

REMARK 3.16
Theorem 3.15 and the above described construction yield a method for factorizing any **L**-concept lattice $\mathcal{B}(G,M,I)$ by any a-cut ${}^{a}E$ of the induced similarity E. It is worth noticing that that the similarity E is defined 'internally', i.e. it is not supplied from the outside.

REMARK 3.17
If **L** the algebra of intuitionistic logic (Heyting algebra) or the algebra of Gödel logic [10] then each a-cut of E is indeed a congruence relation.

Next we formulate a statement concerning the relation of the similarity and the hierarchy of concepts which shows that the further the concepts are in the hierarchy, the less similar they are.

THEOREM 3.18
Let for $\langle A_i, B_i\rangle \in \mathcal{B}(G,M,I)$, $i = 1,2,3$, it holds $\langle A_1, B_1\rangle \leq \langle A_2, B_2\rangle \leq \langle A_3, B_3\rangle$. Then

$$\begin{aligned} E(\langle A_1, B_1\rangle, \langle A_3, B_3\rangle) &\leq E(\langle A_1, B_1\rangle, \langle A_2, B_2\rangle), \\ E(\langle A_1, B_1\rangle, \langle A_3, B_3\rangle) &\leq E(\langle A_2, B_2\rangle, \langle A_3, B_3\rangle). \end{aligned}$$

PROOF. By the assumptions, i.e.$A_1(g) \leq A_2(g) \leq A_3(g)$ for all $g \in G$, and by the antitonicity of $\to$ in the first argument we have $E(\langle A_1, B_1\rangle, \langle A_3, B_3\rangle) = \bigwedge_{g\in G}(A_3(g) \to A_1(g)) \leq \bigwedge_{g\in G}(A_2(g) \to A_1(g)) = E(\langle A_1, B_1\rangle, \langle A_2, B_2\rangle)$. The second part may be obtained symmetrically. ∎

3.4 *Similarity of concept lattices*

Finally, we consider similarity of concept lattices. A natural way to define the similarity degree of two concept lattices over the sets G and M is based on the following intuition. Concept lattices $\mathcal{B}(G,M,I_1)$ and $\mathcal{B}(G,M,I_2)$ are similar iff for each concept $c_1 \in \mathcal{B}(G,M,I_1)$ there is a concept $c_2 \in \mathcal{B}(G,M,I_2)$ such that c_1 and c_2 are similar and, conversely, for each concept $c_2 \in \mathcal{B}(G,M,I_2)$ there is a concept $c_1 \in \mathcal{B}(G,M,I_1)$ such that c_1 and c_2 are similar. In the following we write $\mathcal{B}_1$ and $\mathcal{B}_2$ instead of $\mathcal{B}(G,M,I_1)$ and $\mathcal{B}(G,M,I_2)$, respectively. According to how the similarity of concepts is measured we distinguish two rules for the definition of the similarity degree of two concept lattices:

$$\begin{aligned} &E^*(\mathcal{B}(G,M,I_1), \mathcal{B}(G,M,I_2)) \\ &\quad = \bigwedge_{\langle A_1,B_1\rangle \in \mathcal{B}_1} \bigvee_{\langle A_2,B_2\rangle \in \mathcal{B}_2} E^*(\langle A_1, B_1\rangle, \langle A_2, B_2\rangle) \\ &\qquad \wedge \bigwedge_{\langle A_2,B_2\rangle \in \mathcal{B}_2} \bigvee_{\langle A_1,B_1\rangle \in \mathcal{B}_1} E^*(\langle A_1, B_1\rangle, \langle A_2, B_2\rangle), \end{aligned} \tag{3.21}$$

$* \in \{Ext, Int\}$, where we put

$$\begin{aligned} E^{Ext}(\langle A_1, B_1\rangle, \langle A_2, B_2\rangle) &= E(A_1, A_2) \\ E^{Int}(\langle A_1, B_1\rangle, \langle A_2, B_2\rangle) &= E(B_1, B_2). \end{aligned}$$

Note that E^{Ext} and E^{Int} correspond to the cases when the similarity of concepts is measured by extents and by intents of concepts, respectively. However, Theorem 3.11 cannot be applied to show $E^{Ext} = E^{Int}$ because concepts of different concept lattices are considered.

We are going to show that all of the above relations are in fact similarity relations, that they are equal, and that they are, moreover, equal to the similarity relation E defined on the set of all contexts by

$$E(\langle G, M, I_1\rangle, \langle G, M, I_2\rangle) = E(I_1, I_2) = \bigwedge_{\langle g,m\rangle \in G\times M} I_1(g, m) \leftrightarrow I_2(g, m). \qquad (3.22)$$

The following corollary follows directly from Lemma 3.6.

COROLLARY 3.19
The relation E defined by (3.22) is the largest $\otimes$-similarity relation on $\{\langle G, M, I\rangle \mid I \in L^{G\times M}\}$ such that $I_1(g, m) \otimes E(I_1, I_2) \le I_2(g, m)$ holds for every $g \in G$, $m \in M$.

We need the following lemmata.

LEMMA 3.20
Let $\langle G, M, I_1\rangle$, $\langle G, M, I_2\rangle$ be **L**-contexts, $A \in L^G$, $B \in L^M$. Then $E(I_1, I_2) \le E(A^{\uparrow_{I_1}}, A^{\uparrow_{I_2}})$ and $E(I_1, I_2) \le E(B^{\downarrow_{I_1}}, B^{\downarrow_{I_2}})$.

PROOF. Due to symmetry we prove only the first part, i.e.

$$E(I_1, I_2) \le E(A^{\uparrow_{I_1}}, A^{\uparrow_{I_2}}),$$

i.e.

$$E(I_1, I_2) \le \bigwedge_{m\in M} (A^{\uparrow_{I_1}}(m) \leftrightarrow A^{\uparrow_{I_2}}(m)),$$

which holds iff for each $m \in M$

$$E(I_1, I_2) \le A^{\uparrow_{I_1}}(m) \leftrightarrow A^{\uparrow_{I_2}}(m) = A^{\uparrow_{I_1}}(m) \to A^{\uparrow_{I_2}}(m) \wedge A^{\uparrow_{I_2}}(m) \to A^{\uparrow_{I_1}}(m)$$

holds. The last inequality holds iff $E(I_1, I_2)$ is less than or equal to both $A^{\uparrow_{I_1}}(m) \to A^{\uparrow_{I_2}}(m)$ and $A^{\uparrow_{I_2}}(m) \to A^{\uparrow_{I_1}}(m)$. We check only

$$E(I_1, I_2) \le A^{\uparrow_{I_1}}(m) \to A^{\uparrow_{I_2}}(m),$$

which is equivalent to

$$A^{\uparrow_{I_1}}(m) \otimes E(I_1, I_2) \le A^{\uparrow_{I_2}}(m) = \bigwedge_{g\in G} A(g) \to I_2(g, m)$$

iff for each $g \in G$

$$A^{\uparrow_{I_1}}(m) \otimes E(I_1, I_2) \le A(g) \to I_2(g, m)$$

iff

$$A(g) \otimes A^{\uparrow_{I_1}}(m) \otimes E(I_1, I_2) \leq I_2(g, m).$$

We have

$$\begin{aligned} &A(g) \otimes A^{\uparrow_{I_1}}(m) \otimes E(I_1, I_2) \\ &\quad = \quad A(g) \otimes \bigwedge_{g' \in G} (A(g') \to I_1(g', m)) \otimes \bigwedge_{\langle g', m' \rangle \in G \times M} (I_1(g', m') \leftrightarrow I_2(g', m')) \\ &\quad \leq \quad A(g) \otimes (A(g) \to I_1(g, m)) \otimes (I_1(g, m) \to I_2(g, m)) \leq I_2(g, m) \end{aligned}$$

which had to be proved. ∎

Note that by Lemma 3.8 and Lemma 3.20 we have $E(A_1, A_2) \otimes E(I_1, I_2) \leq E(A_1^{\uparrow_{I_1}}, A_2^{\uparrow_{I_2}})$ and $E(B_1, B_2) \otimes E(I_1, I_2) \leq E(B_1^{\downarrow_{I_1}}, B_2^{\downarrow_{I_2}})$.

LEMMA 3.21
For every **L**-contexts $\langle G, M, I_1 \rangle$, $\langle G, M, I_2 \rangle$ and $* \in \{Ext, Int\}$ it holds

$$E(\langle G, M, I_1 \rangle, \langle G, M, I_2 \rangle) \leq E^*(\mathcal{B}(G, M, I_1), \mathcal{B}(G, M, I_2)).$$

PROOF. We proceed only for E^{Ext}, the second case is symmetric. We have to prove $E(I_1, I_2) \leq E^{Ext}(\mathcal{B}_1, \mathcal{B}_2)$, which holds iff both

$$E(I_1, I_2) \leq \bigwedge_{\langle A_1, B_1 \rangle \in \mathcal{B}_1} \bigvee_{\langle A_2, B_2 \rangle \in \mathcal{B}_2} E^{Ext}(\langle A_1, B_1 \rangle, \langle A_2, B_2 \rangle)$$

and

$$E(I_1, I_2) \leq \bigwedge_{\langle A_2, B_2 \rangle \in \mathcal{B}_2} \bigvee_{\langle A_1, B_1 \rangle \in \mathcal{B}_1} E^{Ext}(\langle A_1, B_1 \rangle, \langle A_2, B_2 \rangle)$$

hold. Due to symmetry we prove only the first inequality which holds iff

$$E(I_1, I_2) \leq \bigvee_{\langle A_2, B_2 \rangle \in \mathcal{B}_2} E(A_1, A_2)$$

holds for each $\langle A_1, B_1 \rangle \in \mathcal{B}_1$. By Lemma 3.20 we have $E(I_1, I_2) \leq E(B_1^{\downarrow_{I_1}}, B_1^{\downarrow_{I_2}}) = E(A_1, B_1^{\downarrow_{I_2}})$, i.e. putting $\langle A_2, B_2 \rangle = \left\langle B_1^{\downarrow_{I_2}}, B_1^{\downarrow_{I_2}\uparrow_{I_2}} \right\rangle$ we have

$$E(I_1, I_2) \leq E(A_1, B_1^{\downarrow_{I_2}}) \leq \bigvee_{\langle A_2, B_2 \rangle \in \mathcal{B}_2} E(A_1, A_2),$$

finishing the proof. ∎

LEMMA 3.22
For every **L**-contexts $\langle G, M, I_1 \rangle$, $\langle G, M, I_2 \rangle$ and $* \in \{Ext, Int\}$ it holds

$$E^*(\mathcal{B}(G, M, I_1), \mathcal{B}(G, M, I_2)) \leq E(\langle G, M, I_1 \rangle, \langle G, M, I_2 \rangle).$$

PROOF. We proceed only for E^{Ext}, the case E^{Int} can be handled analogously. We have to prove

$$E^{Ext}(\mathcal{B}_1,\mathcal{B}_2) \leq E(I_1,I_2) = \bigwedge_{\langle g,m\rangle\in G\times M} I_1(g,m) \leftrightarrow I_2(g,m),$$

which holds iff

$$E^{Ext}(\mathcal{B}_1,\mathcal{B}_2) \leq I_1(g,m) \leftrightarrow I_2(g,m) = (I_1(g,m) \to I_2(g,m)) \wedge (I_2(g,m) \to I_1(g,m))$$

holds for every $g \in G$, $m \in M$. The inequality holds iff both $E^{Ext}(\mathcal{B}_1,\mathcal{B}_2) \leq I_1(g,m) \to I_2(g,m)$ and $E^{Ext}(\mathcal{B}_1,\mathcal{B}_2) \leq I_2(g,m) \to I_1(g,m)$ hold. For symmetry we show only the first inequality which is equivalent to

$$I_1(g,m) \otimes E^{Ext}(\mathcal{B}_1,\mathcal{B}_2) \leq I_2(g,m). \tag{3.23}$$

We have

$$\begin{aligned}
I_1(g,m) \otimes E^{Ext}(\mathcal{B}_1,\mathcal{B}_2) &= I_1(g,m) \\
&\quad \otimes \bigwedge_{\langle A_1,B_1\rangle\in\mathcal{B}_1} \bigvee_{\langle A_2,B_2\rangle\in\mathcal{B}_2} E(A_1,A_2) \wedge \bigwedge_{\langle A_2,B_2\rangle\in\mathcal{B}_2} \bigvee_{\langle A_1,B_1\rangle\in\mathcal{B}_1} E(A_1,A_2) \\
&\leq I_1(g,m) \otimes \bigvee_{\langle A_2,B_2\rangle\in\mathcal{B}_2} E(\{{}^{1}/m\}^{\downarrow_{I_1}},A_2) \wedge \bigvee_{\langle A_1,B_1\rangle\in\mathcal{B}_1} E(A_1,\{{}^{1}/m\}^{\downarrow_{I_2}}) \\
&\leq I_1(g,m) \otimes \bigvee_{\langle A_2,B_2\rangle\in\mathcal{B}_2} E(\{{}^{1}/m\}^{\downarrow_{I_1}},A_2) \\
&\leq I_1(g,m) \otimes \bigvee_{B\in L^M} E(\{{}^{1}/m\}^{\downarrow_{I_1}},B^{\downarrow_{I_2}}) \\
&= \bigvee_{B\in L^M} \big(I_1(g,m) \otimes (\bigwedge_{g'\in G} \{{}^{1}/m\}^{\downarrow_{I_1}}(g') \leftrightarrow B^{\downarrow_{I_2}}(g'))\big) \\
&\leq \bigvee_{B\in L^M} \big(I_1(g,m) \otimes (\{{}^{1}/m\}^{\downarrow_{I_1}}(g) \leftrightarrow (\bigwedge_{m'\in M} B(m') \to I_2(g,m'))\big) \\
&\leq \bigvee_{B\in L^M} \big(I_1(g,m) \otimes (\{{}^{1}/m\}^{\downarrow_{I_1}}(g) \to (\bigwedge_{m'\in M} B(m') \to I_2(g,m'))\big) \\
&\leq \bigvee_{B\in L^M} \big(I_1(g,m) \\
&\quad \otimes((\bigwedge_{m'\in M} \{{}^{1}/m\}(m') \to I_1(g,m')) \to (\bigwedge_{m'\in M} 1 \to I_2(g,m')))\big) \\
&\leq \bigvee_{B\in L^M} \big(I_1(g,m) \otimes (I_1(g,m) \to (\bigwedge_{m'\in M} I_2(g,m'))\big) \\
&\leq \bigvee_{B\in L^M} \big(I_1(g,m) \otimes (I_1(g,m) \to I_2(g,m))\big) \leq I_2(g,m),
\end{aligned}$$

i.e. (3.23) holds. The proof is complete. ■

THEOREM 3.23
For every **L**-contexts $\langle G,M,I_1\rangle$, $\langle G,M,I_2\rangle$ it holds

$$E(\langle G,M,I_1\rangle,\langle G,M,I_2\rangle) = E^{Ext}(\mathcal{B}_1,\mathcal{B}_2) = E^{Int}(\mathcal{B}_1,\mathcal{B}_2).$$

Therefore, E^{Ext} and E^{Int} are similarity relations on $\{\mathcal{B}(G, M, I) \mid I \in L^{G \times M}\}$.

PROOF. The assertion follows immediately by Corollary 3.19, Lemma 3.21, and Lemma 3.22. ∎

Both E^{Ext} and E^{Int} may thus be denoted by E. From the computational point of view, Theorem 3.23 shows that the computation of the similarity of two concept lattices defined naturally by (3.21) may be reduced to the computation of the similarity of the corresponding contexts which is usually much more simple. Indeed, the direct computation of $E(\mathcal{B}_1, \mathcal{B}_2)$ requires $|\mathcal{B}_1| \cdot |\mathcal{B}_2| \cdot (|G| + |M|)$ evaluation of $\leftrightarrow$ while computing $E(I_1, I_2)$ requires $|G| \cdot |M|$ evaluation of $\leftrightarrow$. Note that usually $|G|, |M| \ll |\mathcal{B}(G, M, I)|$ holds.

4 Example

In this section we present an illustrative example. Consider **L** with $L = \{0, \frac{1}{2}, 1\}$ and the Łukasiewicz structure defined on L, cf. Section 2. The context is given by Table 1. The set G contains nine elements (Mercury, ..., Pluto), the set M contains four attributes ('size small', ..., 'near to sun'). The corresponding fuzzy concept lattice is depicted in Figure 1.

TABLE 1. Fuzzy context given by planets and their properties

	size		from sun	
	small (ss)	large (sl)	far (df)	near (dn)
Mercury (Me)	1	0	0	1
Venus (V)	1	0	0	1
Earth (E)	1	0	0	1
Mars (Ma)	1	0	$\frac{1}{2}$	1
Jupiter (J)	0	1	1	$\frac{1}{2}$
Saturn (S)	0	1	1	$\frac{1}{2}$
Uranus (U)	$\frac{1}{2}$	$\frac{1}{2}$	1	0
Neptune (N)	$\frac{1}{2}$	$\frac{1}{2}$	1	0
Pluto (P)	1	0	1	0

To get a deeper insight, the elements (i.e. concepts) of the lattice are identified in Table 2. The similarity relation E^G on G (cf. Theorem 3.4) is shown in Table 3.

Consider now the a-cut of the induced similarity on $\mathcal{B}(G, M, I)$ for $a = \frac{1}{2}$, i.e. ${}^{\frac{1}{2}}E$. The tolerance blocks (which are, in fact, complete sublattices) are depicted in Figure 2. Notice that each block is a maximal subset of **L**-concepts which are similar in the degree at least $\frac{1}{2}$. The corresponding factor lattice $\mathcal{B}(G, M, I)/{}^{\frac{1}{2}}E$ is depicted in Figure 3.

First, note that the concepts which were found depend on the fuzzy context which is given by a subjective judgement (e.g. to what degree we consider 'Mars is far from sun' to be true). Second, there are apparently natural concepts in the concept lattice (e.g. 14 ('small planet near to sun')), as well as concepts which 'were found' (e.g. 26 ('a planet far from sun which is at least partially large')). Concept no. 1 is an example of an empirically empty concept. Concepts which do not contain any element in the degree 1 in their extents (e.g. 1, 2, 3) are partially (empirically) empty.

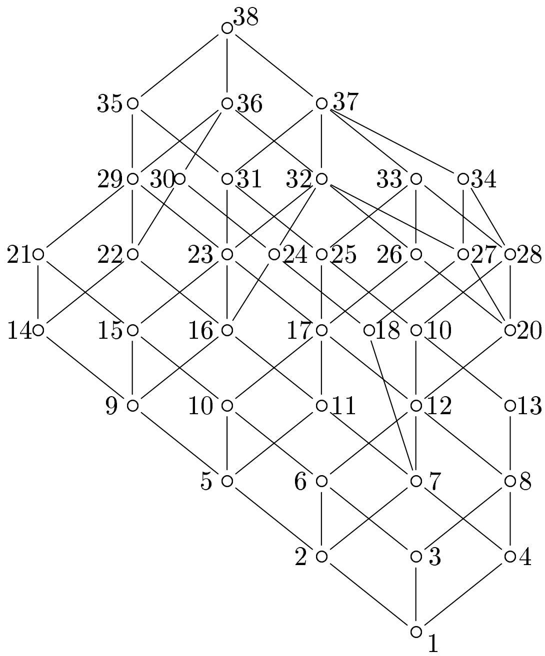

FIGURE 1. Concept lattice of the context in Table 1

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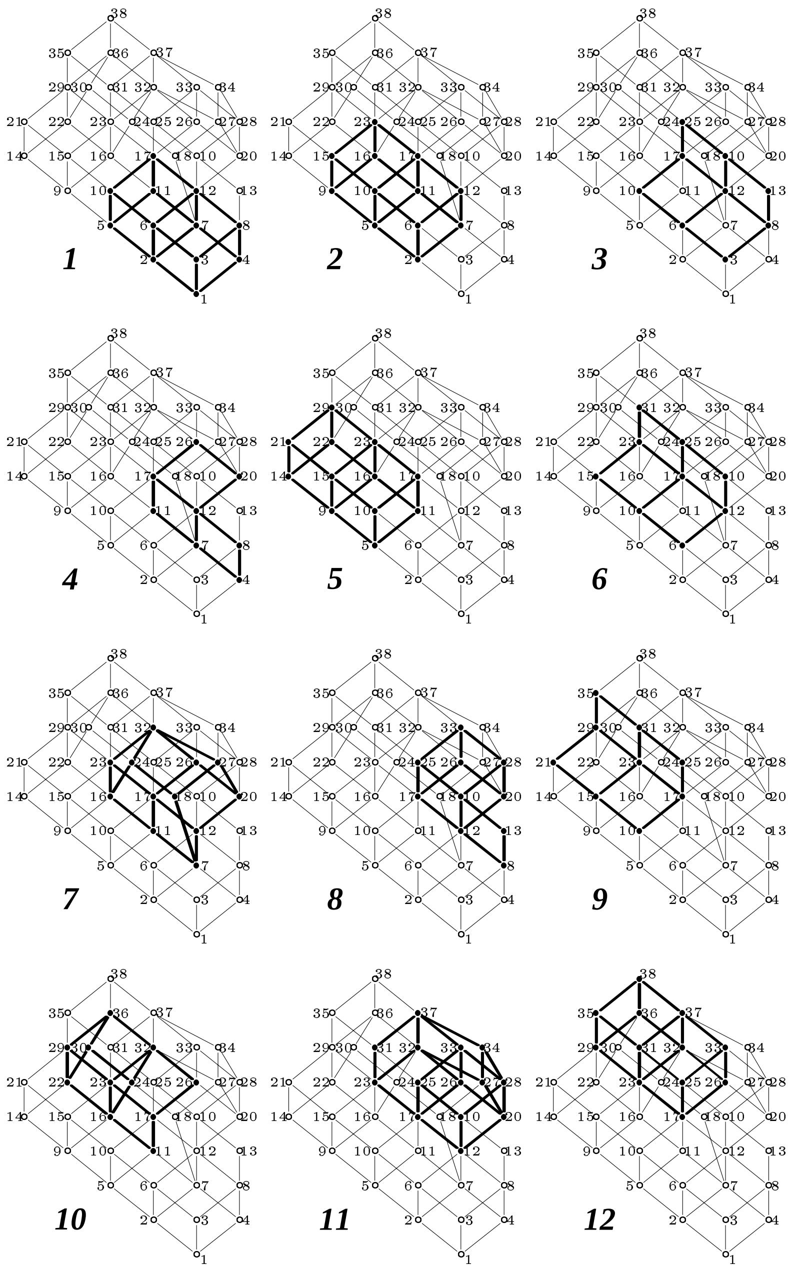

FIGURE 2. Blocks of the tolerance relation $\frac{1}{2}E$ on the concept lattice of Figure 1

TABLE 2. Fuzzy concepts of the context of Table 1

no.	extent									intent			
	Me	V	E	Ma	J	S	U	N	P	ss	sl	df	dn
1.	0	0	0	0	0	0	0	0	0	1	1	1	1
2.	0	0	0	$\frac{1}{2}$	0	0	0	0	0	1	$\frac{1}{2}$	1	1
3.	0	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	$\frac{1}{2}$	1	1	1
4.	0	0	0	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0	1	1	1	$\frac{1}{2}$
5.	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	0	0	1	$\frac{1}{2}$	$\frac{1}{2}$	1
6.	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	1	1
7.	0	0	0	$\frac{1}{2}$	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	1	$\frac{1}{2}$
8.	0	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	1	1	$\frac{1}{2}$
9.	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	0	0	0	0	0	1	0	$\frac{1}{2}$	1
10.	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1
11.	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
12.	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$
13.	0	0	0	0	1	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	1	1	$\frac{1}{2}$
14.	1	1	1	1	0	0	0	0	0	1	0	0	1
15.	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	$\frac{1}{2}$	0	$\frac{1}{2}$	1
16.	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	0	$\frac{1}{2}$	$\frac{1}{2}$
17.	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
18.	0	0	0	$\frac{1}{2}$	0	0	$\frac{1}{2}$	$\frac{1}{2}$	1	1	0	1	0
19.	0	0	0	$\frac{1}{2}$	1	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{1}{2}$
20.	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	0
21.	1	1	1	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	$\frac{1}{2}$	0	0	1
22.	1	1	1	1	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	0	0	$\frac{1}{2}$
23.	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$
24.	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	0	0	$\frac{1}{2}$	$\frac{1}{2}$	1	1	0	$\frac{1}{2}$	0
25.	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
26.	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0
27.	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	1	1	$\frac{1}{2}$	0	1	0
28.	0	0	0	$\frac{1}{2}$	1	1	1	1	$\frac{1}{2}$	0	$\frac{1}{2}$	1	0
29.	1	1	1	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{1}{2}$
30.	1	1	1	1	0	0	$\frac{1}{2}$	$\frac{1}{2}$	1	1	0	0	0
31.	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	1	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{1}{2}$	$\frac{1}{2}$
32.	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	1	1	1	$\frac{1}{2}$	0	$\frac{1}{2}$	0
33.	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	1	1	1	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	0
34.	0	0	0	$\frac{1}{2}$	1	1	1	1	1	0	0	1	0
35.	1	1	1	1	1	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	$\frac{1}{2}$
36.	1	1	1	1	$\frac{1}{2}$	$\frac{1}{2}$	1	1	1	$\frac{1}{2}$	0	0	0
37.	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	1	1	1	1	1	0	0	$\frac{1}{2}$	0
38.	1	1	1	1	1	1	1	1	1	0	0	0	0

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TABLE 3. Similarity relation on objects

	Me	V	E	Ma	J	S	U	N	P
Me	1	1	1	$\frac{1}{2}$	0	0	0	0	0
V		1	1	$\frac{1}{2}$	0	0	0	0	0
E			1	$\frac{1}{2}$	0	0	0	0	0
Ma				1	0	0	0	0	0
J					1	1	$\frac{1}{2}$	$\frac{1}{2}$	0
S						1	$\frac{1}{2}$	$\frac{1}{2}$	0
U							1	1	$\frac{1}{2}$
N								1	$\frac{1}{2}$
P									1

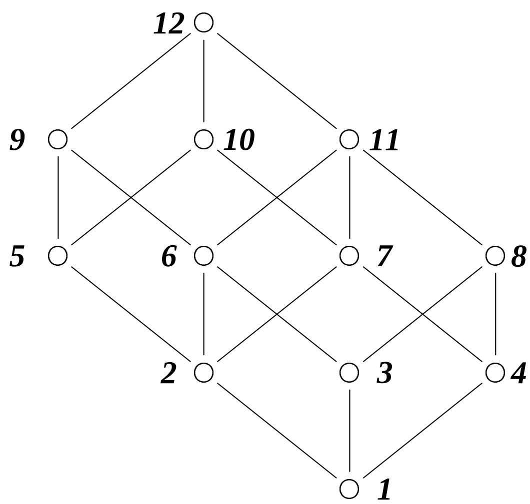

FIGURE 3. Factor lattice $\mathcal{B}(G, M, I)/_{\frac{1}{2}}E$

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