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Sup-t-norm and inf-residuum are one type of relational product: Unifying framework and consequences☆

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Dedicated to the memory of Ladislav J. Kohout

Abstract

We present a simple framework which enables us to consider the well-known sup-t-norm and inf-residuum products of relations as two particular cases of a single, more general type of product. We present basic properties of the framework and consequences for the theory of fuzzy relations. Informally, the paper implies that in many cases of fuzzy relational modeling, such as in solving fuzzy relational equations, there is no need to develop the methods for sup-t-norm and inf-residuum products separately, because these methods are just two particular instances of a single method.

Keywords: Residuated lattice; Fuzzy logic; Fuzzy relations; Relational product; Duality

1. Problem setting

1.1. Two types of relational product

Relational products (called also compositions) are a crucial concept in fuzzy set theory and its applications [25,29]. The so-called max–min product of fuzzy relations was introduced in Zadeh's seminal paper [40] and has played an important role ever since. Further types of products were introduced and extensively studied by Bandler and Kohout [2–7,30,31]. Most important among them are the ∘- and ◁-products of fuzzy relations defined by

$$(R \circ S)(x, z) = \bigvee_{y \in Y} (R(x, y) \otimes S(y, z)), \tag{1}$$

$$(R \triangleleft S)(x, z) = \bigwedge_{y \in Y} (R(x, y) \to S(y, z)). \tag{2}$$

Here, R and S are fuzzy relations between X and Y, and Y and Z, respectively; $(R \circ S)$ and $(R \triangleleft S)$ are new relations between X and Z; $\bigvee$ and $\bigwedge$ denote the supremum and infimum, and $\otimes$ and $\to$ denote the (truth functions of)

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conjunction and implication, respectively, defined on a set L of truth degrees. One reasonable choice is to take the real unit interval [0,1] for L, a left-continuous t-norm for $\otimes$ and its residuum for $\rightarrow$, in which case $\circ$ and $\triangleleft$ are called the sup-t-norm and inf-residuum products. Such choice is a particular case of a more general one in which $\langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ forms a complete residuated lattice (see Section 1.6). Among the many existing studies of fuzzy relations and their products, we mention [24,13,9] which emphasize residuated structures of truth degrees and focus on "graded" properties. Note that

$$(R \triangleright S)(x, z) = \bigwedge_{y \in Y} (S(y, z) \rightarrow R(x, y)) \tag{3}$$

defines another type of product but since $R \triangleright S = (S^{-1} \triangleleft R^{-1})^{-1}$, with $\cdots^{-1}$ denoting the inverse of $\ldots$, we omit $\triangleright$ from our considerations below.

The above-mentioned relational products are employed in a variety of areas that deal with fuzzy relations and their applications [25,29]. Perhaps the best known application of the $\circ$-product is the so-called compositional rule of inference, which is employed in fuzzy controllers [24,29]. Further areas, that are referred to in our paper, include fuzzy relational equations (started in [38], for surveys see [18,19,24,29]) and formal concept analysis (FCA) of data with fuzzy attributes (started in [17] and continued in a number of studies including [37,11,14,33–35]; see also [20] for FCA of data with binary attributes).

1.2. … and two types of product-based models with similar theories

Both the $\circ$- and the $\triangleleft$-product have a distinct, easy-to-understand, meaning. Namely, x is related to z via $(R \circ S)$ if there exists y that is related to both x (via R) and to z (via S), while x is related to z via $(R \triangleleft S)$ if every y that is related to x (via R) is also related to z (via S). Think of a situation where x's, y's, and z's are patients, symptoms, and diseases, and R and S represent relationships "to have a symptom" and "to be a symptom of disease". Then x is related to z via $(R \circ S)$ if patient x has at least one symptom of disease z while x is related to z via $(R \triangleleft S)$ if all the symptoms that patient x has are symptoms of disease z. As a result, two types of models involving products of relations were proposed in the literature, one based on $\circ$ and the other on $\triangleleft$. If one inspects the theories of these models one may notice a remarkable similarity. That is, if one inspects the concepts involved, the theorems, and the proofs of a particular model based on $\circ$, one may notice a similarity to those of the corresponding model based on $\triangleleft$.

For illustration, we present an example regarding fuzzy relational equations and their solvability. Consider the following two types of equations:

$$U \circ S = T, \tag{4}$$

called the sup-t-norm equation, and

$$U \triangleleft S = T, \tag{5}$$

called the inf-residuum equation. In (4) and (5), $S \in L^{Y \times Z}$ and $T \in L^{X \times Z}$ are given fuzzy relations between Y and Z, and X and Z, respectively. One looks for an (unknown) fuzzy relation $U \in L^{X \times Y}$ satisfying (4) and (5). It is well known that if (4) is solvable, $(S \triangleleft T^{-1})^{-1}$ is its largest solution (cf. Corollary 12), and that if (5) is solvable, $T \triangleleft S^{-1}$ is its largest solution (cf. Corollary 13). Even though $(S \triangleleft T^{-1})^{-1}$ and $T \triangleleft S^{-1}$ do not look apparently "dual", the proofs of the results behind these two types of equations are rather similar [24].

1.3. In bivalent case: the two products are mutually reducible

It is well-known that in the bivalent case, i.e. the case of ordinary (crisp) relations, the $\circ$- and $\triangleleft$-products are mutually definable (the argument appears e.g. in [21] where it is present in the context of operators induced by binary relations). Namely, one can check that

$$R \circ S = \overline{R \triangleleft \overline{S}} \quad \text{and} \quad R \triangleleft S = \overline{R \circ \overline{S}}, \tag{6}$$

where $\overline{U}$ denotes a complement of U.

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1.4. In general case: the two types of product are in fact one type

In a complete residuated lattice, however, (6) may not hold because the law of double negation is not satisfied in residuated lattices in general. The situation is described by the following lemma. (A negation of a truth degree a is defined by $\neg a = a \to 0$ and $\overline{U}(x) = \neg U(x)$.)

Lemma 1. *If the structure of truth degrees is a complete residuated lattice then* (6) *being true for every* R *and* S *is equivalent to the law of double negation, i.e. to* $a = \neg\neg a$.

Proof. Note that every residuated lattice satisfying the law of double negation satisfies also

$$\neg\left(\bigvee_{i\in I} a_i\right) = \bigwedge_{i\in I} \neg a_i, \tag{7}$$

$$\neg\left(\bigwedge_{i\in I} a_i\right) = \bigvee_{i\in I} \neg a_i, \tag{8}$$

$$a \to b = \neg(a \otimes \neg b). \tag{9}$$

Namely, (7) holds true in every complete residuated lattice and (8) and (9) are consequences of the law of double negation [13, Eqn. (2.103), (2.104)]. We thus have

$$(R \circ S)(x,z) = \neg\neg((R \circ S)(x,y)) = \neg\left(\neg\left(\bigvee_y R(x,y) \otimes S(y,z)\right)\right)$$
$$= \neg\left(\bigwedge_y \neg(R(x,y) \otimes S(y,z))\right) = \neg\left(\bigwedge_y (R(x,y) \to \neg S(y,z))\right) = \overline{R \lhd \overline{S}}(x,y)$$

and thus also

$$R \lhd S = \overline{\overline{R \lhd \overline{\overline{S}}}} = \overline{R \circ \overline{S}}.$$

Conversely, assume (6) and put $X = Y = Z = \{\omega\}$, $R(\omega,\omega) = 1$ and $S(\omega,\omega) = a$. Then $a = (R \lhd S)(\omega,\omega) = \overline{R \circ \overline{S}}(\omega,\omega) = \neg\neg a$, proving the claim. □

Nevertheless, there exists a simple framework, presented in Section 2, that enables us to see that both $\circ$ and $\lhd$ are in fact two particular cases of a more general type of product.

1.5. Contributions and goals of the paper

The above-mentioned framework allows us to see a particular duality between $\circ$ and $\lhd$. It follows from Section 1.3 that in the bivalent case, to obtain results for $\lhd$, it is enough to establish results for $\circ$ and use the reduction formulas (6), and vice versa. In the general case, however, one may get the results for both $\circ$ and $\lhd$ as a simple consequence of the results for the general type of product.

The presentation of the framework, its basic properties, relationships to existing work, and applications to fuzzy set theory is the main contribution of the present paper. In addition, our goal is to expound consequences for particular areas of fuzzy relational modeling that involve relational products.

1.6. Preliminaries from residuated lattices

A residuated lattice [25,27,39] is an algebra $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \to, 0, 1\rangle$ such that $\langle L, \wedge, \vee, 0, 1\rangle$ is a lattice with 0 and 1 being the least and greatest element of L, respectively; $\langle L, \otimes, 1\rangle$ is a commutative monoid (i.e. $\otimes$ is commutative, associative, and $a \otimes 1 = 1 \otimes a = a$ for each $a \in L$); $\otimes$ and $\to$ satisfy adjointness:

$$a \otimes b \le c \quad \text{iff} \quad a \le b \to c \tag{10}$$

for each $a, b, c \in L$. A residuated lattice is called complete if $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice.

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Residuated lattices play a fundamental role in fuzzy logic and fuzzy set theory [23,25,26]. Elements a of L are called truth degrees (or grades); $\otimes$ and $\to$ are the (truth functions of) conjunction and implication. Examples of residuated lattices are well-known; the most important ones are those with $L = [0, 1]$ (real unit interval), $\wedge$ and $\vee$ being the minimum and maximum, $\otimes$ being a left-continuous t-norm [28] with the corresponding residuum $\to$. A particular case of a residuated lattice is the two-element Boolean algebra $\langle\{0, 1\}, \wedge, \vee, \otimes, \to, 0, 1\rangle$, in which the operations $\wedge, \vee, \otimes, \to$ are the truth functions of the corresponding connectives of classical logic.

Given a residuated lattice $\mathbf{L}$, we define the usual notions [23,25]: an L-set (fuzzy set, graded set) A in a universe U is a mapping $A : U \to L$, $A(u)$ being interpreted as "the degree to which u belongs to A". L^U (or $\mathbf{L}^U$ if it is desirable to make the structure of $\mathbf{L}$ explicit) denotes the collection of all L-sets in U. Binary L-relations (binary fuzzy relations) between U and V can be thought of as L-sets in the universe $U \times V$. For L-sets A and B in universe U, we put

$$A \subseteq B \text{ if and only if } A(u) \leq B(u) \text{ for each } u \in U; \tag{11}$$

in this case, we say that A is included in B.

2. Unifying framework

This section introduces the framework announced in Section 1.5. We intend to keep the framework simple, yet expressive enough to see that $\circ$- and $\lhd$-products are two particular cases of a general type of product and to handle some essential properties of these products. Limitations that result from this simplicity as well as possible extensions of the framework are briefly discussed in Section 4.

2.1. The framework: left-continuous isotone aggregation

Both the $\circ$- and $\lhd$-products share a scheme that involves a function $\square$ which may be thought of as performing aggregation of truth degrees. In case of $\circ$, $\square$ is $\otimes$; in case of $\lhd$, $\square$ is $\to$.

Trying to look at $\otimes$ and $\to$ as two instances of a general "aggregation" might seem strange at first. Namely, $\otimes$ and $\to$ are considered as the (truth functions of) conjunction and implication in fuzzy logic [25,26] and it is well known that these functions have different properties. For example, conjunction is commutative, associative, isotone, etc., while implication does not have any of these properties. However, a kind of duality between $\otimes$ and $\to$, which is based on looking at $\otimes$ and $\to$ as functions with essentially the same properties, is suggested in [13, Theorem 2.20] where it is shown how a residuated lattice may be defined starting from the properties of residuum and adding multiplication as the operation connected to residuum via adjointness.

We assume a general aggregation function $\square : L_1 \times L_2 \to L_3$ with L_1, L_2, and L_3 being sets of grades. In particular, the structure we use is defined as follows.

Definition 1. A *sup-preserving aggregation structure* (aggregation structure, for short) is a quadruple $\langle \mathbf{L}_1, \mathbf{L}_2, \mathbf{L}_3, \square\rangle$ where $\mathbf{L}_i = \langle L_i, \leq_i\rangle$ $(i = 1, 2, 3)$ are complete lattices and $\square : L_1 \times L_2 \to L_3$ is a function which commutes with suprema in both arguments.

Remark 1.

(a) The operations in $\mathbf{L}_i$ are denoted as usual, adding subscript i. That is, the infima, suprema, the least, and the greatest element in $\mathbf{L}_2$ are denoted by $\bigwedge_2$, $\bigvee_2$, 0_2, and 1_2, respectively; the same for $\mathbf{L}_1$ and $\mathbf{L}_3$.

(b) Commuting of $\square$ with suprema in both arguments means that for any $a, a_j \in L_1$ $(j \in J)$, $b, b_{j'} \in L_2$ $(j' \in J')$,

$$\left(\bigvee_{1\, j\in J} a_j\right) \square b = \bigvee_{3\, j\in J} (a_j \square b) \quad \text{and} \quad a \square \left(\bigvee_{2\, j'\in J'} b_{j'}\right) = \bigvee_{3\, j'\in J'} (a \square b_{j'}). \tag{12}$$

(c) Since the supremum of the empty set is the least element, commuting with suprema implies that

$$0_1 \square a_2 = 0_3 \quad \text{and} \quad a_1 \square 0_2 = 0_3, \tag{13}$$

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for every $a_1 \in L_1$ and $a_2 \in L_2$. If commuting with suprema were understood as commuting with suprema of non-empty sets, the situation would be different. This is explained in Example 3. The concept of an aggregation structure with $\Box$ commuting with non-empty suprema is more general and is not considered in this paper.

(d) It follows from the well-known relationship between commuting with suprema and left-continuity (see e.g. [13, Lemma 2.85]) that $\langle\langle[0,1],\leq\rangle,\langle[0,1],\leq\rangle,\langle[0,1],\leq\rangle,\Box\rangle$ is an aggregation structure if and only if the projections $x\mapsto x\Box b$ and $y\mapsto a\Box y$ are non-decreasing left-continuous functions on [0, 1] for which $0\Box b = a\Box 0 = 0$, for all $a, b \in [0,1]$.

(e) We put indices in a_1 and the like for mnemonic reasons. For example, a_1 indicates that a_1 is taken from L_1.

Define operations $\circ_\Box : L_1 \times L_3 \to L_2$ and ${}_\Box\circ : L_3 \times L_2 \to L_1$ (adjoints to $\Box$) by

$$a_1 \circ_\Box a_3 = \bigvee\nolimits_2 \{a_2 | a_1 \Box a_2 \leq_3 a_3\}, \tag{14}$$

$$a_3 {}_\Box\circ a_2 = \bigvee\nolimits_1 \{a_1 | a_1 \Box a_2 \leq_3 a_3\}. \tag{15}$$

Note that due to (13), sets $\{a_2 | a_1\Box a_2 \leq_3 a_3\}$ and $\{a_1 | a_1\Box a_2 \leq_3 a_3\}$ are both non-empty.

For convenience, we sometimes call an aggregation structure the 6-tuple $\langle \mathbf{L}_1, \mathbf{L}_2, \mathbf{L}_3, \Box, \circ_\Box, {}_\Box\circ\rangle$. In our setting, $\langle \mathbf{L}_1, \mathbf{L}_2, \mathbf{L}_3, \Box, \circ_\Box, {}_\Box\circ\rangle$ plays a role analogous to the role of residuated lattices.

Example 1. Let $\langle L, \wedge, \vee, \otimes, \to, 0, 1\rangle$ be a complete residuated lattice with a partial order $\leq$. The following two particular cases, in which $L_i = L$ and $\leq_i$ is either $\leq$ or the dual of $\leq$ (i.e. $\leq_i = \leq$ or $\leq_i = \leq^{-1}$) are important for our purpose.

(a) Let $\mathbf{L}_1 = \langle L, \leq\rangle$, $\mathbf{L}_2 = \langle L, \leq\rangle$, and $\mathbf{L}_3 = \langle L, \leq\rangle$, let $\Box$ be $\otimes$. Then, as is well known from the properties of residuated lattices 39,23], $\Box$ commutes with suprema in both arguments. Furthermore,

$$a_1 \circ_\Box a_3 = \bigvee \{a_2 | a_1 \otimes a_2 \leq a_3\} = a_1 \to a_3$$

and, similarly, $a_3 {}_\Box\circ a_2 = a_2 \to a_3$.

(b) Let $\mathbf{L}_1 = \langle L, \leq\rangle$, $\mathbf{L}_2 = \langle L, \leq^{-1}\rangle$, and $\mathbf{L}_3 = \langle L, \leq^{-1}\rangle$, let $\Box$ be $\to$. Then, $\Box$ commutes with suprema in both arguments. Namely, the conditions (12) for commuting with suprema in this case become

$$\left(\bigvee_{j\in J} a_j\right) \to b = \bigwedge_{j\in J} (a_j \to b) \quad \text{and} \quad a \to \left(\bigwedge_{j\in J} b_j\right) = \bigwedge_{j\in J} (a \to b_j)$$

which are well-known properties of residuated lattices. In this case, we have

$$a_1 \circ_\Box a_3 = \bigwedge \{a_2 | a_1 \to a_2 \geq a_3\} = a_1 \otimes a_3$$

and

$$a_3 {}_\Box\circ a_2 = \bigvee \{a_1 | a_1 \to a_2 \geq a_3\} = a_3 \to a_2.$$

Example 2. Let $L_1 = \{0, 1\}$, $L_2 = [0, 1]$, $L_3 = [0, 1]$, let $\leq_1$, $\leq_2$, $\leq_3$ be the usual total orders on L_1, L_2, and L_3, respectively. Let $\Box$ be defined by $a_1\Box a_2 = \min(a_1, a_2)$. Then $\mathbf{L}_1, \mathbf{L}_2, \mathbf{L}_3$, and $\Box$ satisfy (12). In this case,

$$0 \circ_\Box a = 1, \quad 1 \circ_\Box a = a$$

and

$$a_3 {}_\Box\circ a_2 = \begin{cases} 0 & \text{for } a_2 > a_3, \\ 1 & \text{for } a_2 \leq a_3. \end{cases}$$

A more general version of this example: $L_1 = \{0 = c_0 < c_1 < \cdots < c_p = 1\}$, $L_2 = [0, 1]$, $L_3 = [0, 1]$, let $\leq_1$, $\leq_2$, $\leq_3$ be the usual total orders on L_1, L_2, and L_3, respectively, let $\otimes$ be a left-continuous t-norm and $\to$ its residuum.

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Let $\Box$ be the restriction of $\otimes$ to $L_1 \times [0,1]$. Then $\Box$ obviously satisfies (12) and we have $a_1 \circ_\Box a_3 = a_1 \to a_3$ and $a_3 {}_\Box\circ a_2$ is the largest $c \in L_1$ that is $\leq a_2 \to a_3$. This is a simple example but it enables us to see that the so-called one-sided concept lattices, introduced independently in [10,32], are in fact defined in terms of aggregation structures.

Example 3. Let $L_1 = \{0_1 = 0, 1, \ldots, p = 1_1\}$, $L_2 = \{0_2 = 0, 1, \ldots, q = 1_2\}$, $L_3 = L_1 \times L_2$, $\leq_1$ and $\leq_2$ be the usual total orders, $\leq_3$ be the lexicographic order, i.e. $\langle c, d\rangle \leq_3 \langle c', d'\rangle$ iff $c <_1 c'$ or $c = c'$ and $d \leq_2 d'$. Let $c \Box d = \langle c, d\rangle$. One may check that $\Box$ commutes with non-empty suprema (cf. Remark 1 (c)). However, (13) is not satisfied. For example $0_1 \Box 1 = \langle 0, 1\rangle \neq \langle 0, 0\rangle = 0_3$ and $1 \Box 0_2 = \langle 1, 0\rangle \neq \langle 0, 0\rangle = 0_3$. In this case, it may happen that $\{a_2 | a_1 \Box a_2 \leq_3 a_3\}$ is empty. In such case, one might put $a_1 \circ_\Box a_3 = 0_2$ (which is consistent with letting the supremum of the empty set be equal to the least element). Doing so, we get

$$a_1 \circ_\Box \langle b_1, b_2\rangle = \begin{cases} 1_2 & \text{for } a_1 <_1 b_1, \\ b_2 & \text{for } a_1 = b_1, \\ 0_2 & \text{for } a_1 >_1 b_1. \end{cases}$$

Proceeding analogously for ${}_\Box\circ$, we get

$$\langle b_1, b_2\rangle {}_\Box\circ a_2 = \begin{cases} b_1 & \text{for } a_2 \leq_2 b_2, \\ b_1 - 1 & \text{for } b_1 > 0, a_2 >_2 b_2, \\ 0_1 & \text{for } b_1 = 0, a_2 >_2 b_2. \end{cases}$$

In such case, however, some properties valid for structures commuting with arbitrary suprema are lost. As an example, it is not true in general that $a_2 \leq_2 a_1 \circ_\Box a_3$ implies $a_1 \Box a_2 \leq_3 a_3$ (cf. (16)). To see this, put $a_1 = 2$, $a_2 = 0$, $a_3 = \langle 1, 1\rangle$. Moreover, in this example, we have the following cancellation properties: $c \circ_\Box (c \Box d) = d$ and $(c \Box d) {}_\Box\circ d = c$.

In what follows, we show several properties of $\Box$, $\circ_\Box$, and ${}_\Box\circ$. Most of them are counterparts to well-known properties of residuated lattices. The proofs use standard arguments. Therefore, we present only parts of the proofs with comments. Some properties of residuated lattices do not have their direct counterparts in terms of aggregation structures. For example, $a \Box 1 = a$ (which is a direct transcription of $a \otimes 1 = a$ in the setting of Example 1 (a)) makes no sense because L_1 and L_3 are in general different sets. However, these properties may still have a counterpart in terms of aggregation structures. An example is (20) which is a counterpart to "$a \leq b = 1$ iff $a \leq b$". We also mention some properties of residuated lattices whose direct counterparts in terms of aggregation structures exist but are not true. An example is $1_1 \Box 1_2 = 1_3$ (a counterpart to $1 \otimes 1 = 1$ in the setting of Example 1(a)). Namely, putting $a_1 \Box a_2 = 0_3$ defines an aggregation structure for any $\langle L_i, \leq_i\rangle$'s. In this case, $1_1 \Box 1_2 = 1_3$ is obviously violated.

Theorem 1.

$$a_1 \Box a_2 \leq_3 a_3 \quad \textit{iff} \quad a_2 \leq_2 a_1 \circ_\Box a_3 \quad \textit{iff} \quad a_1 \leq_1 a_3 {}_\Box\circ a_2, \tag{16}$$

$$a_1 \Box (a_1 \circ_\Box a_3) \leq_3 a_3, \quad (a_3 {}_\Box\circ a_2) \Box a_2 \leq_3 a_3, \tag{17}$$

$$a_2 \leq_2 a_1 \circ_\Box (a_1 \Box a_2), \quad a_1 \leq_1 (a_1 \Box a_2) {}_\Box\circ a_2, \tag{18}$$

$$a_1 \leq_1 a_3 {}_\Box\circ (a_1 \circ_\Box a_3), \quad a_2 \leq_2 (a_3 {}_\Box\circ a_2) \circ_\Box a_3, \tag{19}$$

$$a_1 \circ_\Box a_3 = 1_2 \quad \textit{iff} \quad a_1 \Box 1_2 \leq_3 a_3, \quad a_3 {}_\Box\circ a_2 = 1_1 \quad \textit{iff} \quad 1_1 \Box a_2 \leq_3 a_3, \tag{20}$$

$$a_1 \Box a_2 = 0_3 \quad \textit{iff} \quad a_2 \leq_2 a_1 \circ_\Box 0_3 \quad \textit{iff} \quad a_1 \leq_1 0_3 {}_\Box\circ a_2, \tag{21}$$

$$a_1 \circ_\Box 1_3 = 1_2, \quad 1_3 {}_\Box\circ a_2 = 1_1, \tag{22}$$

$$0_1 \circ_\Box a_3 = 1_2, \quad a_3 {}_\Box\circ 0_2 = 1_1, \tag{23}$$

$$a_1 \circ_\Box (a_1 \Box 1_2) = 1_2, \quad (1_1 \Box a_2) {}_\Box\circ a_2 = 1_1. \tag{24}$$

Proof. The proof uses standard arguments involving residuation. We prove only selected parts of the claim for illustration.

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Eq. (16): If $a_1\Box a_2 \le_3 a_3$, then $a_2 \le_2 a_1 \circ_\Box a_3$ follows directly from (14). Conversely, if $a_2 \le_2 a_1 \circ_\Box a_3$, then isotony of $\Box$ and (14) imply $a_1\Box a_2 =\le_3 a_1\Box(a_1\circ_\Box a_3) = a_1\Box\bigvee_{2\, a_1\Box a_2\le_3 a_3} a_2 = \bigvee_{2\, a_1\Box a_2\le_3 a_3}(a_1\Box a_2) \le_3 a_3$. Similarly for ${}_\Box\circ$.

Eq. (24): $a_1\circ_\Box(a_1\Box 1_2) = 1_2$ is equivalent to $1_2 \le_2 a_1\circ_\Box(a_1\Box 1_2)$ which is equivalent (due to (16)) to $a_1\Box 1_2 \le_2 a_1\Box 1_2$; the second part is similar. □

Remark 2.

(a) Let $\mathbf{L}_i$'s and $\Box$ be as in Example 1(a). Then, for instance, (16) says that $a_1 \otimes a_2 \le a_3$ iff $a_2 \le a_1 \to a_3$ iff $a_1 \le a_2 \to a_3$; and the first part of (17) says that $a_1 \otimes (a_1 \to a_3) \le a_3$.

(b) Let $\mathbf{L}_i$'s and $\Box$ be as in Example 1(b). In this case, (16) says that $a_1 \to a_2 \ge a_3$ iff $a_2 \ge a_1 \otimes a_3$ iff $a_1 \le a_3 \to a_2$; and the first part of (17) says that $a_1 \to (a_1 \otimes a_3) \ge a_3$.

(c) Notice that for Example 1(a), (24) become $a_1 \to (a_1 \otimes 1) = 1$ and $(1 \otimes a_2) \to a_2 = 1$ which, in case of residuated lattices, collapse to a single identity $a \to a = 1$. In case of Example 1(b), however, the first identity of (24) becomes $a_1 \otimes (a_1 \to 0) = 0$ while the second one becomes $(1 \to a_2) \to a_2 = 1$, i.e. $a_2 \to a_2 = 1$.

Theorem 2.

(1) $\Box$ *is isotone in both arguments.*

(2) $\circ_\Box$ *is antitone in the first and isotone in the second argument.*

(3) ${}_\Box\circ$ *is isotone in the first and antitone in the second argument.*

Proof. (1) follows from the fact that $\Box$ is distributive w.r.t. suprema. For instance, if $a_1 \le_1 b_1$, then $a_1\Box a_2 \le_2 (a_1\Box a_2) \vee_3 (b_1\Box a_2) = (a_1 \vee_3 b_1)\Box a_2 = b_1\Box a_2$. Antitony of $\circ_\Box$ in the first argument follows from the definition of $\circ_\Box$ and from the isotony of $\Box$; isotony of $\circ_\Box$ in the second argument follows from the definition of $\circ_\Box$. The proof for ${}_\Box\circ$ is similar. □

Theorem 3.

$$a\circ_\Box\left(\bigwedge_{3\, j\in J} c_j\right) = \bigwedge_{2\, j\in J}(a\circ_\Box c_j), \quad \left(\bigvee_{1\, j\in J} a_j\right)\circ_\Box c = \bigwedge_{2\, j\in J}(a_j\circ_\Box c), \tag{25}$$

$$c\,{}_\Box\circ\left(\bigvee_{2\, j\in J} b_j\right) = \bigwedge_{1\, j\in J}(c\,{}_\Box\circ\, b_j), \quad \left(\bigwedge_{3\, j\in J} c_j\right){}_\Box\circ\, b = \bigwedge_{1\, j\in J}(c_j\,{}_\Box\circ\, b), \tag{26}$$

$$a\circ_\Box\left(\bigvee_{3\, j\in J} c_j\right) \ge_2 \bigvee_{2\, j\in J}(a\circ_\Box c_j), \quad \left(\bigwedge_{1\, j\in J} a_j\right)\circ_\Box c \ge_2 \bigvee_{2\, j\in J}(a_j\circ_\Box c), \tag{27}$$

$$c\,{}_\Box\circ\left(\bigwedge_{2\, j\in J} b_j\right) \ge_1 \bigvee_{1\, j\in J}(c\,{}_\Box\circ\, b_j), \quad \left(\bigvee_{3\, j\in J} c_j\right){}_\Box\circ\, b \ge_1 \bigvee_{1\, j\in J}(c_j\,{}_\Box\circ\, b). \tag{28}$$

Proof. The proof involves standard arguments regarding residuation. We prove only the first identity of (25): The "$\le$"-part follows from isotony of $\circ_\Box$ in the second argument. For the "$\ge$"-part, $\bigwedge_{2\, j\in J}(a\circ_\Box c_j) \le_2 a\circ_\Box(\bigwedge_{3\, j\in J} c_j)$ iff $a\Box\bigwedge_{2\, j\in J}(a\circ_\Box c_j) \le_2 \bigwedge_{3\, j\in J} c_j$ iff $a\Box\bigwedge_{2\, j\in J}(a\circ_\Box c_j) \le_2 c_j$ for each $j \in J$ iff $\bigwedge_{2\, j\in J}(a\circ_\Box c_j) \le_2 a\circ_\Box c_j$ for each $j \in J$ which is evidently true. □

Theorem 4.

$$a_1\circ_\Box a_3 \textit{ is the greatest element of } \{a_2 \in L_2 | a_1\Box a_2 \le_3 a_3\}, \tag{29}$$

$$a_1\circ_\Box a_3 \textit{ is the greatest element of } \{a_2 \in L_2 | a_1 \le_1 a_3\,{}_\Box\circ\, a_2\}, \tag{30}$$

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$$a_{3\Box}\circ a_2 \text{ is the greatest element of } \{a_1 \in L_1 | a_1 \Box a_2 \leq_3 a_3\}, \tag{31}$$

$$a_{3\Box}\circ a_2 \text{ is the greatest element of } \{a_1 \in L_1 | a_2 \leq_2 a_1 \circ_\Box a_3\}, \tag{32}$$

$$a_1 \Box a_2 \text{ is the least element of } \{a_3 \in L_3 | a_2 \leq_2 a_1 \circ_\Box a_3\}, \tag{33}$$

$$a_1 \Box a_2 \text{ is the least element of } \{a_3 \in L_3 | a_1 \leq_1 a_{3\Box}\circ a_2\}. \tag{34}$$

Proof. Eq. (29): Due to the first part of (17), $a_1\circ_\Box a_3$ is one of the a_2's satisfying $a_1\Box a_2 \leq_3 a_3$. For any a_2 satisfying $a_1 \Box a_2 \leq_3 a_3$, we have $a_2 \leq_2 a_1\circ_\Box a_3$ due to (16), proving (29). Eq. (30) is a consequence of (29) and (16). The proofs of (31)–(34) are similar. □

Remark 3.

(a) Let $\mathbf{L}_i$'s and □ be as in Example 1(a). Then, for instance, the first part of (25) says that $a \to (\bigwedge_{j\in J} c_j) = \bigwedge_{j\in J}(a \to c_j)$. Eqs. (29) and (33) become the well-known facts that $a \to c$ is the greatest b for which $a \otimes b \leq c$ and that $a \otimes b$ is the least c for which $b \leq a \to c$.

(b) Let $\mathbf{L}_i$'s and □ be as in Example 1(b). In this case, the first part of (25) says that $a \otimes (\bigvee_{j\in J} c_j) = \bigvee_{j\in J}(a \otimes c_j)$.

The following theorem provides alternative definitions of aggregation structures.

Theorem 5. *Let $\mathbf{L}_i = \langle L_i, \leq_i\rangle$ be complete lattices for $i = 1, 2, 3$. The following conditions are equivalent for a function $\Box : L_1 \times L_2 \to L_3$.*

1. *□ commutes with suprema, i.e. $\langle \mathbf{L}_1, \mathbf{L}_2, \mathbf{L}_3, \Box\rangle$ is an aggregation structure.*
2. *□ is isotone in both arguments and the sets $\{b | a_1 \Box b \leq_3 a_3\}$ and $\{a | a \Box a_2 \leq_3 a_3\}$ have greatest elements for every a_1, a_2, a_3.*
3. *There exist functions $\circ_\Box : L_1 \times L_3 \to L_2$ and ${}_\Box\circ : L_3 \times L_2 \to L_1$ which satisfy adjointness w.r.t. □, i.e.* (16).

Proof. "1 ⇒ 2" is established in Theorem 2, (29), and (31). "1 ⇒ 3" is established in (16). We prove "2 ⇒ 3" and "3 ⇒ 1".

"2 ⇒ 3": Let $a_2\circ_\Box a_3$ and $a_{3\Box}\circ a_1$ be defined as the greatest elements of $\{b | a_1 \Box b \leq_3 a_3\}$ and $\{a | a \Box a_2 \leq_3 a_3\}$, respectively. If $a_1 \Box a_3 \leq_3 a_3$ then $a_2 \leq_2 a_1\circ_\Box a_3$ by the definition of $a_1\circ_\Box a_3$. If $a_2 \leq_2 a_1\circ_\Box a_3$, isotony of □ and the definition of $a_1\circ_\Box a_3$ yield $a_1 \Box a_2 \leq_3 a_1 \Box (a_1\circ_\Box a_3) \leq_3 a_3$. Proving the rest of (16) is similar.

"3 ⇒ 1": First we show that □ is isotone in both arguments. Let $a_1 \leq_1 b_1$. Since $b_1 \Box a_2 \leq_3 b_1 \Box a_2$, we get $b_1 \leq_2 (b_1 \Box a_2)_\Box\circ a_2$, hence also $a_1 \leq_2 (b_1 \Box a_2)_\Box\circ a_2$ which yields $a_1 \Box a_2 \leq_3 b_1 \Box a_2$, proving isotony in the first argument. Isotony in the right argument is proved in a similar way. $a \Box (\bigvee_{2j} b_j) \geq_3 \bigvee_{3j}(a \Box b_j)$ follows from the isotony of □. $a \Box (\bigvee_{2j} b_j) \leq_3 \bigvee_{3j}(a \Box b_j)$ is equivalent to $\bigvee_{2j} b_j \leq_2 a\circ_\Box \bigvee_{3j}(a \Box b_j)$. The latter inequality holds true iff $b_j \leq_2 a\circ_\Box \bigvee_{3j}(a \Box b_j)$ for each j which is equivalent to $a \Box b_j \leq_3 \bigvee_{3j}(a \Box b_j)$ being true for each j which is evidently true. The proof is finished. □

Call an aggregation structure $\langle \mathbf{L}_1, \mathbf{L}_2, \mathbf{L}_3, \Box\rangle$ commutative if $a_1 \Box a_2 = a_2 \Box a_1$ for every $a_1, a_2 \in L_1 \cap L_2$. Clearly, if $L_1 = L_2$, we get the ordinary notion of commutativity of □. Recall that a principal ideal in a poset $\langle U, \leq\rangle$ given by $u \in U$ is the set $(u]_\leq = \{v \in U | v \leq u\}$.

Theorem 6. *$\langle \mathbf{L}_1, \mathbf{L}_2, \mathbf{L}_3, \Box\rangle$ is commutative if and only if for every $a \in L_1 \cap L_2$ and $c \in L_3$,*

$$L_1 \cap (a\circ_\Box c]_{\leq_2} = L_2 \cap (c_\Box\circ a]_{\leq_1}. \tag{35}$$

Proof. If $b \in L_1 \cap (a\circ_\Box c]_{\leq_2}$ then $b \in L_2$ and $b \leq_2 a\circ_\Box c$, hence $a \Box b \leq_3 c$ and due to commutativity, $b \Box a \leq_3 c$ from which we get $b \leq_1 c_\Box\circ a$, i.e. $b \in L_2 \cap (c_\Box\circ a]_{\leq_1}$, proving $L_1 \cap (a\circ_\Box c]_{\leq_2} \subseteq L_2 \cap (c_\Box\circ a]_{\leq_1}$; the converse inequality is proved in a similar way.

Assume (35), $a_1, a_2 \in L_1 \cap L_2$, and $c \in L_3$. If $a_1 \Box a_2 \leq_3 c$ then $a_2 \leq_2 a_1\circ_\Box c$, i.e. $a_2 \in L_1 \cap (a_1\circ_\Box c]_{\leq_2}$, hence also $a_2 \in L_2 \cap (c_\Box\circ a_1]_{\leq_1}$, i.e. $a_2 \leq_1 c_\Box\circ a_1$ from which we get $a_2 \Box a_1 \leq_3 c$. This shows that $a_1 \Box a_2$ has the same upper bounds as $a_2 \Box a_1$ and thus proves $a_1 \Box a_2 = a_2 \Box a_1$. □

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The following corollary characterizes commutativity of $\Box$ in terms of coincidence of its left and right residua (note that such condition is known for non-commutative residuated lattices).

Corollary 7. *$\langle \mathbf{L}_1, \mathbf{L}_2, \mathbf{L}_3, \Box\rangle$ with $L_1 = L_2$ is commutative if and only if $\circ_\Box$ coincides with $_\Box\circ$.*

Proof. Immediately from Theorem 6 observing that two elements are equal if and only if their principal ideals are equal. $\Box$

The following theorem shows that for a given residuated structure, any of the three operations, $\Box$, $\circ_\Box$, or $_\Box\circ$, may be considered the "basic aggregation function" and the remaining two as its residua. By $\mathbf{L}_i^{\mathrm{d}}$, we denote the dual of $\mathbf{L}_i$, i.e. $\mathbf{L}_i^{\mathrm{d}} = \langle L_i, \geq_i\rangle$. Moreover, for a function $f : A \times B \to C$, $f^{\mathrm{d}} : B \times A \to C$ denotes a function defined by $f^{\mathrm{d}}(b,a) = f(a,b)$.

Theorem 8. *Let $\mathbf{L} = \langle \mathbf{L}_1, \mathbf{L}_2, \mathbf{L}_3, \Box, \circ_\Box, {}_\Box\circ\rangle$ be an aggregation structure. Then*

1. $\mathbf{L}^{\mathrm{d}} = \langle \mathbf{L}_1, \mathbf{L}_3^{\mathrm{d}}, \mathbf{L}_2^{\mathrm{d}}, \circ_\Box, \Box, {}_\Box\circ^{\mathrm{d}}\rangle$ *is an aggregation structure.*
2. ${}^{\mathrm{d}}\mathbf{L} = \langle \mathbf{L}_3^{\mathrm{d}}, \mathbf{L}_2, \mathbf{L}_1^{\mathrm{d}}, {}_\Box\circ, \circ_\Box^{\mathrm{d}}, \Box\rangle$ *is an aggregation structure.*

Proof. By a direct application of Theorem 4 and elementary considerations. $\Box$

Remark 4.
(a) If $\langle \mathbf{L}_1, \mathbf{L}_2, \mathbf{L}_3, \Box, \circ_\Box, {}_\Box\circ\rangle$ is set as in Example 1(a), then $\langle \mathbf{L}_1, \mathbf{L}_3^{\mathrm{d}}, \mathbf{L}_2^{\mathrm{d}}, \circ_\Box, \Box, {}_\Box\circ^{\mathrm{d}}\rangle$ is just the aggregation structure from Example 1(b).
(b) A particular consequence of Theorem 8 is mentioned in Remark 5.
(c) It is easily seen that $(\mathbf{L}^{\mathrm{d}})^{\mathrm{d}} = \mathbf{L}$ and ${}^{\mathrm{d}}({}^{\mathrm{d}}\mathbf{L}) = \mathbf{L}$.

2.2. Historical notes

In this section, we present an account of related work. The basic motivation behind aggregation structures is to have a framework within which the (truth functions of) conjunction $\otimes$ and its residuated implication $\to$ have similar properties. As far as the author knows, a kind of this duality was suggested for the first time in [13, Theorem 2.20] but this line of thought has not been developed further in [13].

Another related work, with quite different motivations, is [33,16]. In [33] Krajči developed a common generalization of two approaches to formal concept analysis (FCA) of data with fuzzy attributes, namely one presented e.g. in [11–13,37] and the other presented in [32,10]. In doing so, he utilized a structure consisting of three sets of truth degrees with certain isotone aggregation function. This approach looked different from the "mainstream approach" that is based on residuated structures of truth degrees. It has been shown in [16, Theorem 5] that the structure with three sets of truth degrees used in [33] is in fact a kind of a three-sorted residuated structure based on certain monotone aggregation function, called tentatively a "residuated structure for generalized concept lattices" in [16]. The theorem is given without proof in [16] with a reference for the proof to an extended version of the paper. The extended version was never submitted and part of the present paper, namely the equivalence of conditions 1 and 3 of Theorem 5 (which is essentially the content of [16, Theorem 5]), may be considered as filling the gap by providing the proof. However, the three-sorted structure itself has not been investigated in [16]. The work from [33] has recently been continued by Medina et al., see e.g. [35], who developed an approach to FCA with fuzzy attributes based on the so-called multi-adjoint structures of truth degrees. Again, the structure of truth degrees utilized in that approach is not studied in [35] itself. Note also that neither of [16,33,35] mentions the possibility of obtaining two particular types of concept lattices, namely one when the aggregation function is conjunction and the other when the aggregation is implication (cf. Example 1(a) and (b)). This possibility is observed in [15] where the concept of aggregation structure is utilized to provide a common framework for optimal decompositions of matrices with entries from residuated lattices. The present paper is a continuation of [15].

Let us provide more detailed comments on the structure of truth degrees used in [35] (we keep the notation from [35]). The basic notion involved is that of an adjoint triple. Let $\langle P_i, \leq_i\rangle$ $(i = 1, 2, 3)$ be partially ordered sets and

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$\& : P_1 \times P_2 \to P_3$, $\swarrow: P_3 \times P_2 \to P_1$, and $\nwarrow: P_3 \times P_1 \to P_2$ be functions. Then $\langle \&, \swarrow, \nwarrow \rangle$ is called an adjoint triple if

1. & is isotone in both arguments;
2. $\swarrow$ and $\nwarrow$ are isotone in the first and antitone in the second argument;
3. the following form of adjointness holds for any $x \in P_1$, $y \in P_2$, $z \in P_3$:

$$x \leq_1 z \swarrow y \quad \text{iff} \quad x \& y \leq_3 z \quad \text{iff} \quad y \leq_2 z \nwarrow x. \tag{36}$$

Let us first mention that this definition of an adjoint triple is redundant. Namely:

Lemma 2. $\langle \&, \swarrow, \nwarrow \rangle$ *forms an adjoint triple if and only if* (36) *holds.*

Proof. We need to show the monotony conditions from 1 and 2 of the above definition. One may check that the argument showing isotony of $\Box$ in the proof of "3 ⇒ 1" of Theorem 5 is independent of whether the underlying posets are complete lattices and thus proves isotony of &. Isotony of $\swarrow$ and $\nwarrow$ in the first argument may be proved similarly. Let $b_2 \leq_2 b_1$. As $c \swarrow b_1 \leq_1 c \swarrow b_1$, we get $(c \swarrow b_1) \& b_1 \leq_3 c$, hence $b_1 \leq_2 c \nwarrow (c \swarrow b_1)$ due to (36). Therefore, $b_2 \leq_2 c \nwarrow (c \swarrow b_1)$ from which we get $c \swarrow b_1 \leq_1 c \swarrow b_2$ by two applications of (36), proving anitony of $\swarrow$ in the second argument. The case of $\nwarrow$ is proved similarly. □

The structure used in [35], called a multi-adjoint frame, consists essentially of a collection of adjoint triples in which $\langle P_1, \leq_1 \rangle$ and $\langle P_2, \leq_2 \rangle$ are complete lattices. These structures are exactly the "residuated structure for generalized concept lattices" from [16] mentioned above. Note also that in [35], the authors mention that there is a condition missing in [16, Theorem], namely that the aggregation function preserves least elements, i.e. condition (13). However, this is not correct because (13) is true as a consequence of commuting with suprema (see Remark 1 (c)) and thus need not be mentioned in [16, Theorem].

Another paper related to the present one is [22]. The authors show that antitone fuzzy Galois connections studied in [12] and their natural counterparts, isotone fuzzy Galois connections, which are introduced in [22], have a common generalization. For this purpose, they introduce a structure consisting of five complete lattices equipped with two functions that satisfy certain conditions regarding injectivity of their projections and their compatibility with lattice operations. Obtaining isotone Galois connections from antitone ones is accomplished by flipping certain lattices which is similar to the approach presented in this paper.

Another related stream of research are the various studies of systems of logic connectives, see e.g. [25]. Particularly relevant to the present paper is the work by Morsi et al., on associatively tied implications, see e.g. [1,36]. In [1], the authors study so-called implication triples which consist of three (truth functions of) connectives of conjunction and two implications defined on a single partially ordered set with a greatest element that are related by adjointness of the form (16) which satisfy some additional properties. As in the case of [35], the authors' definition of an implication triple is redundant because the authors overlooked that the monotony conditions of the connectives are entailed by adjointness. Later [36], the authors extended this approach by involving two partially ordered sets in the notion of an implication triple. The concept of an aggregation structure is more general than this concept of an implication triple. The concept of an implication triple is itself employed in the notion of a tied adjointness algebra (10-tuple containing an implication triple and a so-called residuation algebra which is "tied" to the implication triple via a condition generalizing $(a \otimes b) \to c = a \to (b \to c)$). It is to be noted, however, that the idea of taking duals of partially ordered sets to obtain new implication triples from given ones which leads to a certain duality principle of the propositional calculus developed in [36] is present in that paper. A more detailed overview of and comparison to this work is beyond the scope of this paper.

3. Consequences: pairs of classic concepts and results as pairs of instances of a general case

In this section, we show that the $\circ$ and $\lhd$ products are one type of product if developed within the framework of aggregation structures, and provide, by means of examples, further consequences of developing fuzzy relational models in this framework. As is mentioned in Section 2.2, further examples, related to decompositions of matrices, are to appear in [15].

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3.1. $\circ$ (sup-t-norm) and $\lhd$ (inf-residuum) are one type of product

Let $\langle \mathbf{L}_1, \mathbf{L}_2, \mathbf{L}_3, \Box\rangle$ be an aggregation structure. Let $\boxdot$, $\lhd_\Box$, and ${}_\Box\lhd$ be relational products defined as follows:

- For $\rho \in L_1^{U\times V}$ and $\sigma \in L_2^{V\times W}$, let $\rho \boxdot \sigma \in L_3^{U\times W}$ be defined by
$$(\rho \boxdot \sigma)(u, w) = \bigvee_{v\in V}{}_3(\rho(u, v)\Box\sigma(v, w)) \tag{37}$$
for every $u \in U$ and $w \in W$.
- For $\rho \in L_1^{U\times V}$ and $\sigma \in L_3^{V\times W}$, let $\rho\lhd_\Box\sigma \in L_2^{U\times W}$ be defined by
$$(\rho\lhd_\Box\sigma)(u, w) = \bigwedge_{v\in V}{}_2(\rho(u, v)\circ_\Box\sigma(v, w)) \tag{38}$$
for every $u \in U$ and $w \in W$.
- For $\rho \in L_2^{U\times V}$ and $\sigma \in L_3^{V\times W}$, let $\rho_\Box \lhd\sigma \in L_1^{U\times W}$ be defined by
$$(\rho_\Box \lhd\sigma)(u, w) = \bigwedge_{v\in V}{}_1(\sigma(v, w)_\Box\circ\rho(u, v)) \tag{39}$$
for every $u \in U$ and $w \in W$.

The following example justifies the claim from the title of this paper.

Example 4 (*sup-t-norm and inf-residuum are one type of product*). (a) One may easily check that for the setting of Example 1(a),
$$\rho \boxdot \sigma = \rho \circ \sigma \quad \text{(furthermore, } \rho\lhd_\Box\sigma = \rho \lhd \sigma,\ \rho_\Box \lhd\sigma = \rho \lhd \sigma).$$
(b) For the setting of Example 1(b),
$$\rho \boxdot \sigma = \rho \lhd \sigma \quad \text{(furthermore, } \rho\lhd_\Box\sigma = \rho \circ \sigma,\ \rho_\Box \lhd\sigma = (\sigma^{-1} \lhd \rho^{-1})^{-1} = \rho \rhd \sigma).$$
For instance,
$$\begin{aligned}(\rho \boxdot \sigma)(u, w) &= \bigvee_{v\in V}{}_3(\rho(u, v)\Box\sigma(v, w))\\ &= \bigwedge_{v\in V}(\rho(u, v) \to \sigma(v, w)) = \rho \lhd \sigma.\end{aligned}$$

Remark 5. In view of Theorem 8, in order to establish properties of $\lhd_\Box$ and ${}_\Box\lhd$ within an aggregation structure $\langle \mathbf{L}_1, \mathbf{L}_2, \mathbf{L}_3, \Box, \circ_\Box, {}_\Box\circ\rangle$, there is no need to provide direct proofs for them. Namely, they can be easily obtained from the properties of $\Box$. For example, the properties of $\lhd_\Box$ within an aggregation structure $\langle \mathbf{L}_1, \mathbf{L}_2, \mathbf{L}_3, \Box, \circ_\Box, {}_\Box\circ\rangle$ are just the properties of $\boxdot$ within the aggregation structure $\langle \mathbf{L}_1, \mathbf{L}_3^{\mathrm{d}}, \mathbf{L}_2^{\mathrm{d}}, \circ_\Box, \Box, {}_\Box\circ^{\mathrm{d}}\rangle$. We illustrate this remark by the following results, provided here as an example.

Theorem 9. *For any aggregation structure and fuzzy relations $R, R_i \in L_1^{X\times Y}$ and $S, S_i \in L_2^{Y\times Z}$ we have*
$$\left(\bigcup_{i\in I}{}_1 R_i\right) \boxdot S = \bigcup_{i\in I}{}_3(R_i \boxdot S), \quad R \boxdot \left(\bigcup_{i\in I}{}_2 S_i\right) = \bigcup_{i\in I}{}_3(R \boxdot S_i). \tag{40}$$

Proof. Follows directly from the definition of $\boxdot$ and (12). □

Corollary 10. *For any aggregation structure and fuzzy relations $R, R_i \in L_1^{X\times Y}$ and $S, S_i \in L_3^{Y\times Z}$ we have*
$$\left(\bigcup_{i\in I}{}_1 R_i\right) \lhd_\Box S = \bigcap_{i\in I}{}_2(R_i\lhd_\Box S), \quad R\lhd_\Box \left(\bigcap_{i\in I}{}_3 S_i\right) = \bigcap_{i\in I}{}_2(R\lhd_\Box S_i). \tag{41}$$

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Proof. Follows directly from Theorem 9, observing that the $\lhd_\Box$-product in aggregation structure $\langle \mathbf{L}_1, \mathbf{L}_2, \mathbf{L}_3, \Box, \circ_\Box, {}_\Box\circ \rangle$ is just the $\boxdot$-product in aggregation structure $\langle \mathbf{L}_1, \mathbf{L}_3^{\mathrm{d}}, \mathbf{L}_2^{\mathrm{d}}, \circ_\Box, \Box, {}_\Box\circ^{\mathrm{d}} \rangle$, cf. Theorem 8. □

Therefore, we get the well-known results about $\circ$-, $\lhd$- and $\rhd$-products and their commutativity with unions and intersections, such as $(\bigcup_{i\in I} R_i) \circ S = \bigcup_{i\in I}(R_i \circ S)$, $R \circ (\bigcup_{i\in I} S_i) = \bigcup_{i\in I}(R \circ S_i)$, $(\bigcup_{i\in I} R_i) \lhd S = \bigcap_{i\in I}(R_i \lhd S)$, $R \lhd (\bigcap_{i\in I} S_i) = \bigcap_{i\in I}(R \lhd S_i)$, $(\bigcap_{i\in I} R_i) \rhd S = \bigcap_{i\in I}(R_i \rhd S)$, $R \rhd (\bigcup_{i\in I} S_i) = \bigcap_{i\in I}(R \rhd S_i)$, as corollaries of (40).

3.2. Fuzzy relational equations

As the second illustration, we consider the following relational equation. Let $S \in L_2^{Y\times Z}$, $T \in L_3^{X\times Z}$, i.e. S and T are fuzzy relations between Y and Z, and X and Z, respectively. We look for a fuzzy relation $U \in L_1^{X\times Z}$ for which

$$U \boxdot S = T. \tag{42}$$

Theorem 11. *Eq. (42) is solvable if and only if $(S_\Box \lhd T^{-1})^{-1}$ is its solution. Moreover, if (42) is solvable, $(S_\Box \lhd T^{-1})^{-1}$ is its largest solution.*

Proof. Let (42) be solvable. If R a solution then

$$\bigvee_{y\in Y}{}_3 (R(x,y) \Box S(y,z)) = T(x,z)$$

for every $x \in X$, $z \in Z$, hence

$$R(x,y) \Box S(y,z) \le_3 T(x,z)$$

for every $y \in Y$, which is equivalent to

$$R(x,y) \le_1 T(x,z) {}_\Box\circ S(y,z)$$

due to (16). Therefore,

$$R(x,y) \le_1 \bigwedge_{z\in Z}{}_1 (T(x,z) {}_\Box\circ S(y,z)) = \bigwedge_{z\in Z}{}_1 (T^{-1}(z,x) {}_\Box\circ S(y,z)) = (S_\Box \lhd T^{-1})^{-1}(x,y).$$

We proved

$$R \subseteq_1 (S_\Box \lhd T^{-1})^{-1}, \tag{43}$$

i.e. that any solution R is smaller than $(S_\Box \lhd T^{-1})^{-1}$. Due to isotony of $\Box$, (43) implies $R \boxdot S \subseteq_3 (S_\Box \lhd T^{-1})^{-1} \boxdot S$. Furthermore, one may easily verify that $(S_\Box \lhd T^{-1})^{-1} \boxdot S \subseteq_3 T$. Hence,

$$T = R \boxdot S \subseteq_3 (S_\Box \lhd T^{-1})^{-1} \boxdot S \subseteq_3 T,$$

showing that if R is a solution then $(S_\Box \lhd T^{-1})^{-1}$ is a solution as well. □

Let us now observe that the well-known results on solvability of (4) and (5) are simple consequences of Theorem 11.

Corollary 12. *Eq. (4) is solvable if and only if $(S \lhd T^{-1})^{-1}$ is its solution. Moreover, if (4) is solvable, $(S \lhd T^{-1})^{-1}$ is its largest solution.*

Proof. If $\mathbf{L}_i$'s and $\Box$ are as in Example 1(a), (42) becomes (4). Example 4(a) implies that $(S_\Box \lhd T^{-1})^{-1} = (S \lhd T^{-1})^{-1}$. Since $\subseteq_1$ coincides with $\subseteq$, the assertion is a particular instance of Theorem 11. □

Corollary 13. *Eq. (5) is solvable if and only if $T \lhd S^{-1}$ is its solution. Moreover, if (5) is solvable, $T \lhd S^{-1}$ is its largest solution.*

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Proof. If $\mathbf{L}_i$'s and $\square$ are as in Example 1(b), (42) becomes (5). Example 4(b) implies that $(S_\square \lhd T^{-1})^{-1} = (((T^{-1})^{-1} \lhd S^{-1})^{-1})^{-1} = (T \lhd S^{-1})$. Since $\subseteq_1$ coincides with $\subseteq$, the assertion is again a particular instance of Theorem 11. $\square$

Remark 6. For aggregation structures other than those from Example 1(a) and (b), Theorem 11 provides solutions to other, new type of fuzzy relational equations. For example, consider the aggregation structure from Example 2 with $L_1 = \{0 = c_0 < c_1 < \cdots < c_p = 1\}$, $L_2 = [0, 1]$, $L_3 = [0, 1]$. In this case, Theorem 11 provides results regarding solutions of fuzzy relational equations that may be described as follows. Given fuzzy relations $S \in [0, 1]^{Y\times Z}$ and $T \in [0, 1]^{X\times Z}$, find a fuzzy relation $U \in [0, 1]^{X\times Y}$ that satisfies $U \circ S = T$ and, at the same time, uses only truth degrees from L_1. Thus, if $L_1 = \{0, \frac{1}{2}, 1\}$, one looks for fuzzy relations constrained by the requirement that any two elements are not related at all, are half-related, or fully related.

4. Conclusions and further topics

In this paper, it is shown that the $\circ$ and $\lhd$ products of fuzzy relations, known also as the sup-t-norm and inf-residuum products, are two particular cases of a more general type of product. For this purpose, a simple unifying framework is developed which is based on a binary function commuting with suprema. Basic properties of this framework are established and some applications in fuzzy relational modeling are provided as examples.

Future research shall include the following topics. First, applications of this framework to the theory of fuzzy sets, including a more thorough studies of the ones mentioned in this paper, are to be explored. Note that the generalizations within the framework presented in this paper, e.g. of the theorem describing solutions of fuzzy relational equations, are more or less straightforward, given the established results, e.g. those for the particular cases of fuzzy relational equations. However, this need not be the case, as is shown in [15], where proofs of certain properties of formal concepts established in the framework of residuated lattices use properties that are not available in the framework of aggregation structures; see also the paragraph preceding Theorem 1.

Second, further variants of the notion of aggregation structure need to be considered. Even though aggregation structures make it possible to generalize important parts of fuzzy relational modeling (e.g. those from fuzzy relational equations outlined in this paper and in [8], those from decomposition of matrices with truth degrees shown in [15], as well as those from formal concept analysis shown in [15]), certain aspects cannot be expressed directly in terms of aggregation structures. An example of this sort is graded inclusion of fuzzy sets. Another is represented by terms involving multiple occurrences of logical connectives such as associativity. In both of these examples, the problem is that the arguments of the connectives from an aggregation structure are taken from different sets of truth degrees. One solution is to consider aggregation structures with the same sets of truth degrees. A different one is to consider additional properties connecting the possibly different sets of truth degrees.

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