CENTRAL POINTS AND APPROXIMATION IN RESIDUATED LATTICES

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Abstract. The paper presents results on approximation in residuated lattices given that closeness is assessed by means of biresiduum. We describe central points and optimal central points of subsets of residuated lattices and provide properties of these points. In addition, we present algorithms for two problems regarding optimal approximation.

1. Introduction and preliminaries

Suppose there is a picture with two poles with lengths \( \leq 1 \). If we do not know the magnification factor of the picture, an obvious way to define a similarity \( s_{1,2} \) of the two poles is to measure their lengths \( a_1 \) and \( a_2 \) in the picture and put

\[
s_{1,2} = \min \left( \frac{a_1}{a_2}, \frac{a_2}{a_1} \right),
\]

because it is independent of the magnification factor of the picture, i.e., \( s_{1,2} = \min \left( \frac{c a_1}{c a_2}, \frac{c a_2}{c a_1} \right) \) for every \( c > 0 \). A “central pole”, i.e., a pole equally similar to each of the poles, then has the length

\[
l = \sqrt{a_1} \cdot \sqrt{a_2}.
\]

If, however, the magnification factor is known, we may obtain the actual lengths \( a_1 \) and \( a_2 \) of the poles and set

\[
s_{1,2} = 1 - |a_1 - a_2|.
\]

In this case, the length \( a \) of the “central pole” is

\[
a = \frac{a_1 + a_2}{2}.
\]

Clearly, for a collection of poles with lengths \( a_i \), the length of the central poles for \( s_{1,2} \) given by (1.1) and (1.3) is

\[
a = \sqrt{\min_i a_i} \cdot \sqrt{\max_i a_i} \quad \text{and} \quad a = \frac{\min_i a_i + \max_i a_i}{2},
\]

respectively.

The above considerations may be carried over to a general setting of approximation in complete residuated lattices [1, 8], i.e., structures \( L = (L, \otimes, \rightarrow, \wedge, \vee, 0, 1) \) such that \( (L, \wedge, \vee, 0, 1) \) is a complete lattice, \( (L, \otimes, 1) \) is a commutative monoid, and \( \otimes \) (multiplication) and \( \rightarrow \) (residuum) form an adjoint pair, i.e., \( a \otimes b \leq c \) iff
In what follows, we use the following three well-known examples of complete residuated lattices on $L = [0, 1]$ induced by continuous $t$-norms [7]: standard Lukasiewicz algebra ($a \otimes b = \max(0, a + b - 1)$, $a \rightarrow b = \min(1, 1 - a + b)$), standard Goguen algebra ($a \otimes b = a \cdot b$, $a \rightarrow b = 1$ if $a \leq b$, and $a \rightarrow b = b/a$ otherwise), standard Gödel algebra ($a \otimes b = \min(a, b)$, $a \rightarrow b = 1$ if $a \leq b$, and $a \rightarrow b = b$ otherwise). One may easily check that in (1.1),

$$s_{1, 2} = a_1 \leftrightarrow a_2$$

with $\leftrightarrow$ being the biresiduum corresponding to the standard Goguen algebra and that in (1.2),

$$a = (a_1 \land a_2) \otimes \sqrt{a_1 \leftrightarrow a_2},$$

with $\sqrt{\phantom{a}}$ denoting the square root of $\otimes$ (see [5] and Section 3). Likewise, (1.5) and (1.6) become (1.3) and (1.4) if $\leftrightarrow$ and $\otimes$ are the biresiduum and $t$-norm of the standard Lukasiewicz algebra.

It is well known [4] that a biresiduum $\leftrightarrow$, defined in any residuated lattice by

$$a \leftrightarrow b = (a \rightarrow b) \land (b \rightarrow a),$$

satisfies

$$a \leftrightarrow b = 1 \quad \text{iff} \quad a = b,$$

$$a \leftrightarrow b = b \leftrightarrow a,$$

$$a \leftrightarrow b \otimes (b \leftrightarrow c) \leq a \leftrightarrow c.$$ 

Hence, $a \leftrightarrow b$ may be interpreted as an element in $L$ representing similarity (closeness) of $a$ and $b$. Note also that (1.7)–(1.9) resemble dual versions of the axioms of a metric with a generalized triangular inequality. Indeed, for the standard Lukasiewicz algebra, $d_{\leftrightarrow}(a, b) = 1 - (a \leftrightarrow b)$ is a $[0, 1]$-valued metric on $[0, 1]$ (more generally, using $d_{\leftrightarrow}(a, b) = f(a \leftrightarrow b)$ one obtains a metric from a biresiduum of a continuous Archimedean $t$-norm with an additive generator $f$ [7]). For the standard Gödel algebra, in which case the $t$-norm is not Archimedean, $d_{\leftrightarrow}(a, b) = 1 - (a \leftrightarrow b)$ is a $[0, 1]$-valued ultrametric on $[0, 1]$. Note also that residuated lattices and their generalizations, developed initially within the studies of ring ideals [8] and since then studied within partially ordered algebraic structures [6], are the main structures of truth degrees used in many-valued logic [2, 3, 4] in which biresiduum is the truth function of the connective of equivalence.

In this paper, we present results motivated by the following problem, alluded to by the initial example. Given a set of elements of a residuated lattice, what are its central points, i.e., elements which are similar (as much as possible or to some specified similarity level) to every element of the set, provided similarity is assessed by means of biresiduum?

We assume familiarity with basic properties of residuated lattices and basic concepts from tolerance relations, i.e., reflexive and symmetric binary relations. A block of a tolerance $T$ on a set $U$ is a subset $B$ of $U$ for which $B \times B \subseteq T$. A maximal block of $T$ is a block $B$ of $T$ which is maximal with respect to set inclusion. A collection of maximal tolerance blocks of $T$ is denoted by $U/T$ and forms a covering of $U$. A class of $T$ given by $u \in U$ is the set $[u]_T = \{v \in U \mid u T v\}$. Clearly, if $T$ is an equivalence, maximal blocks as well as classes of $T$ are just equivalence classes of $T$. 

$$a \leq b \rightarrow c.$$
Throughout the paper, \( L \) denotes a complete residuated lattice and \( e \) an element of its support set \( L \). By \( \approx_e \), we denote the tolerance on \( L \) defined by
\[
a \approx_e b \quad \text{iff} \quad a \leftrightarrow b \geq e.
\]

2. Central points, central sets, and maximal blocks

For \( B \subseteq L \), we set
\[
(2.1) \quad C_e(B) = \{ c \in L \mid \text{for each } b \in B, c \leftrightarrow b \geq e \}.
\]
We call \( C_e(B) \) the \( e \)-central set of \( B \) and its elements \( e \)-central points of \( B \).

**Lemma 2.1.** \( c \in C_e(B) \) iff \((c \to \bigwedge B) \land (\bigvee B \to c) \geq e\).

**Proof.** From \( c \to (\bigwedge_{b \in B} b) = \bigwedge_{b \in B} (c \to b) \) and \((\bigvee_{b \in B} b) \to c = \bigwedge_{b \in B} (b \to c)\).

Denoting by \([p, q]\) the interval \( \{ x \in L \mid p \leq x \leq q \} \subseteq L \), we get the following theorem.

**Theorem 2.2.** \( C_e(B) = [e \otimes \bigvee B, e \to \bigwedge B] \).

**Proof.** By adjointness, \( e \leq c \to \bigwedge B \) is equivalent to \( c \leq e \to \bigwedge B \) and \( e \leq \bigvee B \to c \) is equivalent to \( e \otimes \bigvee B \leq c \). The assertion thus follows from Lemma 2.1.

For \( c \in L \), we let
\[
(2.2) \quad B_e(c) = \{ b \in L \mid c \leftrightarrow b \geq e \}
\]
and call \( B_e(c) \) the closed ball with center \( c \) and radius \( e \). \( B_e(c) \) is called maximal if there is no \( c' \neq c \) such that \( B_e(c) \subseteq B_e(c') \).

**Example 2.3.** In the standard Lukasiewicz algebra, \( B_e(c) = [c - (1 - e), c + (1 - e)] \cap [0, 1] \). The closed ball \( B_{0.5}(0) = [0, 0.5] \) is not maximal, since \( B_{0.5}(0) \subseteq B_{0.5}(0.5) = [0, 1] \).

Note that a closed ball \( B_e(c) \) is just the class of tolerance \( \approx_e \) determined by \( c \). In addition, \( C_e(B) = \bigcap_{c \in B} B_e(c) \) and \( B_e(c) = C_e(\{c\}) \). The following assertion is thus a corollary of Theorem 2.2.

**Theorem 2.4.** \( B_e(c) = [e \otimes c, e \to c] \).

Let for \( a \in L \),
\[
a_e = e \otimes a, \quad a^e = e \to a, \quad [a]_e = [a_e, (a_e)^e], \quad [a]^e = [(a^e)_e, a^e].
\]
Using standard properties of residuated lattices, it is easy to verify that \( \approx_e \) is a compatible tolerance relation on the complete lattice \( \langle L, \leq \rangle \), i.e., \( a_i \approx_e b_i \) \( (i \in I) \) implies \( \bigwedge_{i \in I} a_i \approx_e \bigwedge_{i \in I} b_i \) and \( \bigvee_{i \in I} a_i \approx_e \bigvee_{i \in I} b_i \). Since Theorem 2.4 says that \( a_e \) and \( a^e \) are the least and the greatest elements of \( L \) which are \( \approx_e \)-related to \( a \), \([9]\) yields the following description of maximal blocks of \( \approx_e \).

**Theorem 2.5.** \( L/\approx_e = \{ [a]_e \mid a \in L \} = \{ [a]^e \mid a \in L \} \).

The next lemma shows further properties of balls.

**Lemma 2.6.** \( [c]^e \cap [c]_e = [(c^e)_e, (c_e)^e] \) is the set of all \( d \) for which \( B_e(d) \supseteq B_e(c) \).

Moreover,
\[
(2.3) \quad B_e((c^e)_e) = [e \otimes e \otimes (e \to c), e \to c],
\]
\[
(2.4) \quad B_e((c_e)^e) = [e \otimes c, e \to (e \otimes (e \to c))].
\]
Proof. According to Theorem 2.4, $B_e(d) \supseteq B_e(c)$ is equivalent to $e \otimes d \leq e \otimes c$ and $e \rightarrow c \leq e \rightarrow d$. The first inequality is equivalent to $d \leq e \rightarrow (e \otimes c) = (c_e)^e$, the second one to $(c_e)_e = e \otimes (e \rightarrow c) \leq d$, proving the first part. (2.3) and (2.4) easily follows from Theorem 2.4, and from $e \rightarrow c = e \rightarrow (e \otimes (e \rightarrow c))$, $e \otimes c = e \otimes (e \rightarrow (e \otimes c))$. □

In general, it may happen that if $B_e(c)$ is not maximal, there exists a maximal ball $B_e(d)$ such that $(c_e)_e < d < (c_e)_e$ (i.e., $B_e(d) \supseteq B_e(c)$), $e \otimes d < e \otimes c$ (smallest element of $B_e(d)$ < smallest element of $B_e(c)$), and $e \rightarrow d > e \rightarrow c$ (largest element of $B_e(d)$ > largest element of $B_e(c)$). This is shown in the following example.

Example 2.7. Let $L$ be the Cartesian product of two standard Lukasiewicz algebras on $[0,1]$, $e = \langle 0.7, 0.7 \rangle$, $c = \langle 0.1, 0.9 \rangle$, and $d = \langle 0.2, 0.8 \rangle$. Then $(c_e)_e = \langle 0.1, 0.7 \rangle$, $(c_e)_e = \langle 0.1, 0.9 \rangle$, $B_e(c) = [0,0.6), (0.4,1]$, $B_e(d) = [0,0.5), (0.5,1)]$

This, however, does not happen in linearly ordered residuated lattices where $(c_e)_e$ and $(c_e)_e$ play an important role in describing the maximal balls containing $B_e(c)$.

Theorem 2.8. Let $L$ be linearly ordered, let $B_e(d) \supseteq B_e(c)$.

1. $e \otimes d = e \otimes c$ or $e \rightarrow d = e \rightarrow c$, i.e., $B_e(d)$ and $B_e(c)$ have the same lower boundary or the same upper boundary.

2. If $B_e(d)$ is maximal then $B_e(d) = B_e((c_e)_e)$ in which case $d \in [e \otimes (e \rightarrow c) \rightarrow (e \otimes c(e \rightarrow c))]$, or $B_e(d) = B_e((c_e)_e)$ in which case $d \in [e \otimes (e \rightarrow ((e \otimes c) \rightarrow e \otimes c)) \rightarrow (e \otimes c)]$.

3. If $(c_e)_e$ is maximal then $B_e((c_e)_e) = B_e(c)$. If $(c_e)_e$ is maximal then $B_e((c_e)_e)$ is maximal or $B_e((c_e)_e) = B_e(c)$.

Proof. 1. Since $B_e(d) \supseteq B_e(c)$, $e \otimes d \leq e \otimes c$ and $e \rightarrow d \geq e \rightarrow c$ by Theorem 2.4. Due to linearity, $e \otimes d < e \otimes c$ implies $d < c$ and $e \rightarrow d > e \rightarrow c$ implies $d > c$, hence the claim.

2. According to 1., $e \otimes d = e \otimes c$ or $e \rightarrow d = e \rightarrow c$. Assume $e \otimes d = e \otimes c$ (the proof similar for $e \rightarrow d = e \rightarrow c$). Lemma 2.6 implies $d \leq (c_e)_e$, hence $e \rightarrow d \leq e \rightarrow (c_e)_e = e \rightarrow (e \rightarrow (e \otimes c))$. Therefore, $B_e(d) = [e \otimes c, e \rightarrow d] \subseteq [e \otimes c, e \rightarrow (e \rightarrow (e \otimes c))] = B_e((c_e)_e)$. Maximality of $B_e(d)$ now implies $B_e(d) = B_e((c_e)_e)$. The fact that $d \in [e \otimes (e \rightarrow ((e \otimes c) \rightarrow e \otimes c)) \rightarrow (e \otimes c)]$ follows directly from Lemma 2.6 and from $((c_e)_e)_e = (c_e)_e$.

3. Let $B_e((c_e)_e)$ not be maximal. Then $B_e(c) \subseteq B_e((c_e)_e) \subseteq B_e(d)$ for some maximal $B_e(d)$. According to 2., $B_e(d) = B_e((c_e)_e)$. From (2.3) and (2.4) we get $B_e((c_e)_e) = [e \otimes e \otimes (e \rightarrow c), e \rightarrow c] \subseteq [e \otimes c, e \rightarrow (e \otimes (e \rightarrow c))] = B_e((c_e)_e)$, i.e., $e \otimes c \leq e \otimes e \otimes (e \rightarrow c)$. Since $e \otimes c \geq e \otimes e \otimes (e \rightarrow c)$ is always the case, we conclude $e \otimes c = e \otimes e \otimes (e \rightarrow c)$ which means $B_e((c_e)_e) = [e \otimes c, e \rightarrow c] = B_e(c)$. □

We now turn our attention to the relationship between closed balls with radius $e$ and blocks of the tolerance $\approx_{e^2}$ (where $e^2 = e \otimes e$) and show that maximal closed balls with radius $e$ coincide with maximal blocks of this tolerance. It is easy to check that $B \subseteq B'$ implies $C_e(B) \supseteq C_e(B')$ and that $B \subseteq C_e(C_e(B))$ for every $B \subseteq L$. As a consequence, taking into account that $B_e(c) = C_e((c_e))$, we get the following lemma.

Lemma 2.9. 1. For any $B \subseteq L$, $B \subseteq B_e(c)$ for every $c \in C_e(B)$. For any $c \in L$, $c \in C_e(B_e(c))$. 

2. $C_e(C_e(B))$ is the largest subset of $L$ which has the same $e$-central points as $B$.

**Lemma 2.10.** $C_e(B)$ is non-empty if and only if $B$ is a block of $\approx_{e,2}$. Hence, $B_e(c)$ is a block of $\approx_{e,2}$ for each $c \in B$.

**Proof.** According to Theorem 2.2, $C_e(B)$ is non-empty iff $e \otimes \bigvee B \leq e \rightarrow \bigwedge B$ which is equivalent to $e \otimes e \leq \bigvee B \rightarrow \bigwedge B$. Since $\bigvee B \rightarrow \bigwedge B = \bigwedge_{a,b \in B} a \rightarrow b = \bigwedge_{a,b \in B} a \leftrightarrow b$, we conclude that $C_e(B)$ is non-empty iff $e \otimes e \leq a \leftrightarrow b$ for every $a, b \in B$, proving the claim. As $c \in B_e(c)$, $B_e(c)$ is non-empty and since $B_e(c) = C_e(\{c\})$, $B_e(c)$ is a block of $\approx_{e,2}$. □

**Theorem 2.11.** The following conditions are equivalent.

1. $B$ is a maximal closed ball with radius $e$.
2. $B$ is a maximal block of $\approx_{e,2}$.
3. $C_e(B) \neq \emptyset$ and $C_e(B) = \{c \in L | B = B_e(c)\}$.

**Proof.** “1. $\Rightarrow 2.$”: According to Lemma 2.10, a maximal closed ball $B_e(c)$ is a block of $\approx_{e,2}$. Therefore, $B_e(c) \subseteq B$ for some maximal block $B$ of $\approx_{e,2}$. From Lemma 2.10 it follows that $C_e(B)$ is non-empty. Let thus $c' \in C_e(B)$. Lemma 2.9 implies $B \subseteq B_e(c')$, hence $B_e(c) \subseteq B \subseteq B_e(c')$. Maximality of $B_e(c)$ thus yields $B_e(c) = B$.

“2. $\Rightarrow 3.$” $C_e(B)$ is non-empty due to Lemma 2.10. On the one hand, if $c \in C_e(B)$ then, due to Lemma 2.10 and Lemma 2.9, $B_e(c)$ is a block of $\approx_{e,2}$ and $B \subseteq B_e(c)$. Since $B$ is maximal, we conclude $B = B_e(c)$. On the other hand, if $B = B_e(c)$ then clearly $c \in C_e(B)$.

“3. $\Rightarrow 1.$”: Suppose $B = B_e(c)$ is not maximal. Then $B \subseteq B_e(c')$ for some $c'$. As a consequence, $c' \in C_e(B)$, i.e., $B = B_e(c')$, a contradiction. □

### 3. Optimal central points

An **optimal central point** of $B \subseteq L$ is an element $c \in L$ such that for every $d \in L$,

$$\bigwedge_{a \in B}(a \leftrightarrow d) \leq \bigwedge_{a \in B}(a \leftrightarrow c).$$

That is, the infimum of similarity degrees $a \leftrightarrow c$ of $a \in L$ to $c$ is the largest possible. Since for any $d$, $\bigwedge_{a \in B}(a \leftrightarrow d) = (d \rightarrow \bigwedge B) \land (\bigvee B \rightarrow d)$ (see, for example, the proof of Lemma 2.1), $c$ is an optimal central point iff for every $d \in L$,

$$\bigwedge_{a \in B}(a \leftrightarrow d) \leq (c \rightarrow \bigwedge B) \land (\bigvee B \rightarrow c)$$

We say that $e \in L$ is

- an **admissible radius** of $B$ if $C_e(B) \neq \emptyset$;
- a **radius** of $B$ for $a \in L$ if $e$ is the largest element for which $a \in C_e(B)$.

One can see that the radius of $B$ for $a$ equals $\bigwedge_{a' \in B}(a' \leftrightarrow a)$.

**Theorem 3.1.** Conditions 1., 2., and 3. are equivalent.

1. The set of all optimal central points of $B$ is non-empty and $e$ is the radius of $B$ for some optimal central point $c$ of $B$.
2. The set of all optimal central points of $B$ is non-empty and $e$ is the radius of $B$ for any of the optimal central points.
3. $e$ is the largest admissible radius of $B$.

Any of conditions 1., 2., and 3. implies condition 4.

4. The set of all optimal central points is equal to $C_e(B)$. 

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**CENTRAL POINTS AND APPROXIMATION IN RESIDUATED LATTICES**

5
Proof. “1. ⇒ 2.”: (3.1) implies that the radii of B for any two optimal central points $c_1$ and $c_2$ are equal.

“2. ⇒ 3.”: Clearly, $e$ is an admissible radius of $B$. If $e'$ is an admissible radius of $B$ then for any $d \in C_{e'}(B)$, we have $e' \leq \bigvee_{a \in B}(a \leftrightarrow d)$. For any optimal central point $c$ of $B$, the assumption yields $\bigwedge_{a \in B}(a \leftrightarrow c) = e$. Therefore, (3.1) implies $\bigwedge_{a \in B}(a \leftrightarrow d) \leq e$, whence $e' \leq e$, proving 3.

“3. ⇒ 1.”: For $c \in C_{e}(B)$, $e \leq \bigwedge_{a \in B}(a \leftrightarrow c)$. On the other hand, since $\bigwedge_{a \in B}(a \leftrightarrow c)$ is an admissible radius (the radius of $B$ for $c$), we have $\bigwedge_{a \in B}(a \leftrightarrow c) \leq e$, whence $\bigwedge_{a \in B}(a \leftrightarrow c) = e$. Since for any $d$, $\bigwedge_{a \in B}(a \leftrightarrow d)$ is an admissible radius, we get $\bigwedge_{a \in B}(a \leftrightarrow d) \leq e = \bigwedge_{a \in B}(a \leftrightarrow c)$. Therefore, $e$ is an optimal central point of $B$, proving 1.

“2. ⇒ 4.”: Clearly, every optimal central point of $B$ is in $C_e(B)$. If $d$ is not optimal then $\prod_{a \in B}(a \leftrightarrow d) \neq e$ and hence $d \notin C_e(B)$.

Remark 3.2. A Note that $e$ being the largest admissible radius of $B$ is equivalent to the following condition:

(3.3) for every $c', e' \in L : B \subseteq B_{e'}(c')$ implies $e' \leq e$.

Indeed, if (3.3) is the case and $e'$ is an admissible radius of $B$ then $\emptyset \neq C_{e'}(B)$ and thus for $c' \in C_{e'}(B)$ we have $B \subseteq B_{e'}(c')$ and hence $e' \leq e$ by (3.3). On the other hand, if $e$ is the largest admissible radius of $B$ and $B \subseteq B_{e'}(c')$ then $c' \in C_{e'}(B)$, i.e., $e'$ is an admissible radius of $B$, whence $e' \leq e$.

B. Condition 4. of Theorem 3.1 does not imply conditions 1., 2., and 3. Indeed, consider the standard product algebra on $L = [0, 1]$. The set of optimal central points of $B = \{0\}$ equals $B$. Now, $C_e(B) = B$ for every $a \in (0, 1]$ but the largest admissible radius of $B$ is 0.5.

Theorem 3.3. If $e$ is the largest admissible radius of $B$ then $C_e(C_e(B))$ is the largest subset of $L$ which has the same set of optimal central points as $B$ and for which $e$ is an admissible radius.

Proof. According to 2. of Lemma 2.9, $C_e(C_e(B))$ is the largest subset whose set of $e$-central points is $C_e(B)$. Hence, since $C_e(B) \neq \emptyset$, $e$ is an admissible radius of $C_e(C_e(B)).$ If $e'$ is an admissible radius of $C_e(C_e(B))$ then since $B \subseteq C_e(C_e(B))$, we get $\emptyset \neq C_{e'}(C_e(C_e(B))) \subseteq C_{e'}(B)$, i.e., $e'$ is an admissible radius of $B$, whence $e' \leq e$. Thus, $e$ is the largest admissible radius of $C_e(C_e(B))$ and Theorem 3.1 implies that $C_e(C_e(B))$ has the same optimal central points as $B$. Let $B'$ be another set with the same optimal central points as $B$ for which $e$ is an admissible radius and let $e'$ be the largest admissible radius of $B'$, Then $C_{e'}(B) = C_{e'}(B')$ and, since $e \leq e'$, $C_{e'}(B') \subseteq C_{e'}(B')$. Hence $C_{e'}(B) \subseteq C_{e'}(B')$ from which we get $C_{e}(C_{e}(B)) \supseteq C_{e}(C_{e}(B')) \supseteq B'$ by Lemma 2.9, completing the proof.

Lemma 3.4. 1. $e$ is an admissible radius of $B$ iff $e \otimes e \leq \bigvee B \rightarrow \bigwedge B$.

2. For any $a \in L$, $e = e \wedge (a \rightarrow (\bigvee B \rightarrow \bigwedge B))$ is an admissible radius of $B$.

3. $e$ is an admissible radius of $B$ iff $e = e \wedge (e \rightarrow (\bigvee B \rightarrow \bigwedge B))$.

4. 

(3.4) $R = \{a \wedge (a \rightarrow (\bigvee B \rightarrow \bigwedge B)) \mid a \in L\}$ is the set of all admissible radii of $B$.

Proof. 1. According to Theorem 2.2, $C_{e}(B) \neq \emptyset$ iff $[e \otimes \bigvee B, e \rightarrow \bigwedge B] \neq \emptyset$ iff $e \otimes \bigvee B \leq e \rightarrow \bigwedge B$ iff $e \otimes e \leq \bigvee B \rightarrow \bigwedge B$. 


2. Putting \( d = \sqrt{B} \to \bigwedge B \), we have 
\[ (a \wedge (a \to d)) \otimes (a \wedge (a \to d)) \leq (a \wedge (a \to d)) \otimes (a \wedge (a \to d)) \leq a \otimes (a \to d) \leq d, \]

hence the claim follows from 1.

3. From 1. and adjointness we get that \( e \) is an admissible radius of \( B \) iff \( e \leq e \to (\sqrt{B} \to \bigwedge B) \), which is equivalent to \( e = e \wedge (e \to (\sqrt{B} \to \bigwedge B)) \).

4. A consequence of 2. and 3.

**Theorem 3.5 (optimal central points).** \( B \) has optimal central points if and only if the set \( R \) from (3.4) has a largest element. This element is the largest admissible radius \( e \) of \( B \) and the set of optimal central points of \( B \) equals \( C_\epsilon(B) \).

**Proof.** Follows directly from 4. of Lemma 3.4 and Theorem 3.1.

**Remark 3.6.** The following observations concern the relationship between optimal central points and centers of maximal balls.

A. Let \( B \) have optimal central points and let \( e \) be the corresponding largest admissible radius of \( B \). Then for some optimal central point \( c \) of \( B \), \( B_c(c) \) is a maximal ball with radius \( e \). Indeed, in this case \( C_\epsilon(B) \) is the set of optimal central points and for any maximal ball \( B_c(c) \supseteq B \) (such maximal balls exist because for any \( d \in C_\epsilon(B) \), \( B_d(d) \supseteq B \) is contained in some maximal \( B_c(c) \) ), we have \( c \in C_\epsilon(B(c)) \subseteq C_\epsilon(B) \).

B. However, it may be the case that for an optimal central point \( c \) of \( B = B_c(c) \) with the largest admissible radius \( e \), \( B_c(c) \) is not a maximal ball with radius \( e \). Namely, consider the Lukasiewicz operations on \( L = \{0,\frac{1}{4},\frac{2}{4},\frac{3}{4},1\} \), and let \( c = \frac{3}{4} \), \( e = \frac{2}{4} \). Then \( c \) is an optimal central point of \( B_c(c) = \{\frac{1}{4},\frac{2}{4},\frac{3}{4},1\} \) and \( e \) is the largest admissible radius. However, \( B_c(c) \) is not maximal since \( B_\epsilon(c) \supseteq B_\epsilon(\frac{2}{4}) = L \).

C. Neither is it true that if \( B_e(e) \) is a maximal ball with radius \( e \) then \( c \) is an optimal central point of \( B_e(e) \). Consider the complete residuated lattice given by chain \( L = 0 < a < b < 1 \) equipped with the following operations.

<table>
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<tr>
<th>⊗</th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>1</th>
<th>→</th>
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\( B_a(1) = \{a, b, 1\} \) is a maximal ball since \( B_a(1) = B_a(b) = B_a(a) \) and \( B_a(0) = \{0\} \). However, 1 is not an optimal central point of \( B_a(1) \). Namely, the only optimal central point of \( B_a(1) \) is \( b \) because \( \bigwedge_{x \in B_a(1)}(x \leftrightarrow 1) = \bigwedge_{x \in B_a(1)}(x \leftrightarrow a) = a < b = \bigwedge_{x \in B_a(1)}(x \leftrightarrow b) \).

D. The example from C. also shows that if \( B_c(c) \) is a maximal ball with radius \( e \) and \( c \) is an optimal central point of \( B_c(c) \), \( e \) need not be the largest admissible radius of \( B_c(c) \). Namely, for \( c = b \) and \( e = a \), the largest admissible radius of \( B_c(c) = \{a, b, 1\} \) is \( b \) which is larger than \( e \).

As the next theorem shows, every set in a linearly ordered residuated lattices has optimal central points.

**Theorem 3.7.** Let \( L \) be linearly ordered, \( B \subseteq L \). The set \( R \) of admissible radii of \( B \) is closed under suprema. Hence, \( R \) has a largest element, the largest admissible radius of \( B \), and \( B \) has optimal central points.

**Proof.** Let \( \{e_i \mid i \in I\} \subseteq R \). By 1. of Lemma 3.4, \( e_i \otimes e_i \leq \bigvee B \to \bigwedge B \) for every \( i \in I \). Since \( L \) is linearly ordered, \( e_i \otimes e_j \leq (e_i \otimes e_i) \lor (e_j \otimes e_j) \) for every \( i, j \in I \).
Therefore, $(\bigvee_{i \in I} e_i) \otimes (\bigvee_{i \in I} e_i) = \bigvee_{i,j \in I} e_i \otimes e_j = \bigvee_{i \in I} e_i \otimes e_i \leq \bigvee B \to \bigwedge B$, proving $\bigvee_{i \in I} e_i \in R$, i.e., $R$ is closed under suprema. Since $0 \in R$, $R$ has a largest element.

Remark 3.8. There exists a residuated lattice $L$ and subset $B \subseteq L$ without optimal central points. Let $L = \{0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, 1\}$ be equipped with the ordering specified by the Hasse diagram in Fig. 1 and operations $\otimes$ and $\to$ on $L$ defined as follows.

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Set $B = [a_3, 1]$. The set $R$ from (3.4) of all admissible radii of $B$ is equal to

![Graph](image)

**Figure 1.** Ordered set $L$ from Remark 3.8.

$\{0, a_1, a_2, a_3, a_4, a_5, a_6\}$ and does not have a greatest element. Therefore, there is no optimal radius and, consequently, no optimal central point of $B$. 

Next we derive an explicit description of largest admissible radii and optimal central points for residuated lattices with square roots [5]: A complete residuated lattice $L$ has square roots if there is a function $\sqrt{} : L \to L$ satisfying

\begin{align}
\sqrt{a} \otimes \sqrt{a} &= a, \\
b \otimes b \leq a &\implies b \leq \sqrt{a},
\end{align}

for every $a, b \in L$. As an example, the standard Lukasiewicz, product, and Gödel algebras on $[0,1]$ have square roots. These are given by

- $\sqrt{a} = \frac{a + 1}{2}$ for Lukasiewicz,
- $\sqrt{a} = \text{ordinary number-theoretic square root}$ of $a$ for product,
- $\sqrt{a} = a$ for Gödel.

**Theorem 3.9.** If $L$ has square roots then every subset $B \subseteq L$ has optimal central points. For the corresponding largest admissible radius $e$ of $B$ it holds

$$e = \sqrt{\bigvee B \to \bigwedge B}.$$  

**Proof.** According to 1. of Lemma 3.4, (3.5), and (3.6), $e$ is the largest admissible radius of $B$. The rest follows from Theorem 3.1. \hfill \Box

In general, the existence of optimal central points does not imply the existence of square roots. As an example, for $L = \{0, 0.5, 1\}$ equipped with Lukasiewicz operations, every $B \subseteq L$ has optimal central points. However, $0.5$ does not have a square root.

**Theorem 3.10.** $L$ has square roots iff for every $a \in L$, $[a, 1]$ has an optimal central point such that for the corresponding admissible radius $e$ we have $e \otimes e = a$.

**Proof.** If $L$ has square roots, then according to Theorem 3.9, the largest admissible radius of $[a, 1]$ is $e = \sqrt{1 \to a} = \sqrt{a}$ and we have $\sqrt{a} \otimes \sqrt{a} = a$. Conversely, it follows from the definitions and Theorem 3.1 that the admissible radius corresponding to an optimal central point of $[a, 1]$ is the square root of $a$. \hfill \Box

**Corollary 3.11.** If for every $a \in L$, $[a, 1]$ has an optimal central point such that for the corresponding admissible radius $e$ we have $e \otimes e = a$, then every $B \subseteq L$ has optimal central points.

4. **Optimal approximation algorithms**

We now consider the following type of problems. Given a set $M \subseteq L$, find a small set $K \subseteq L$ which approximates $M$. The degree $\text{appr}(M, K)$ to which $M \subseteq L$ is approximated by $K \subseteq L$ is defined by

$$\text{appr}(M, K) = \bigwedge_{a \in M} \bigvee_{b \in K} (a \leftrightarrow b).$$

$\text{appr}(M, K)$ can be seen as a truth degree of “for every $a \in M$ there is $b \in K$ such that $a$ and $b$ are similar”. Clearly, $\text{appr}(M, K) = 1$ for $M \subseteq K$, and $K_1 \subseteq K_2$ implies $\text{appr}(M, K_1) \leq \text{appr}(M, K_2)$. Consider the following problems.

**Problem 1** Given a finite $M \subseteq L$ and a threshold $e \in L$, find $K \subseteq L$ such that
1. $K$ approximates $M$ to degree at least $e$, i.e.,
\[
\text{appr}(M, K) \geq e,
\]
2. there is no $K'$ with $|K'| < |K|$ for which $\text{appr}(M, K') \geq e$.

**Problem 2** Given a finite $M \subseteq L$ and a threshold $e \in L$, find $K \subseteq L$ satisfying (1) and (2) of Problem 1, and
3. For any $K'$ with $|K'| = |K|$,
\[
\text{appr}(M, K) \geq \text{appr}(M, K'),
\]
\(\text{i.e., among the sets with } |K| \text{ elements, } K \text{ provides the best approximation of } M.\)

In the rest of this section, we assume that the complete residuated lattice $L$ is linearly ordered, i.e., $a \leq b$ or $b \leq a$ for every $a, b \in L$. The following theorem provides a universal description of sets $K$ satisfying (4.2).

**Theorem 4.1.** 1. Let $\Omega \subseteq 2^L$ be a covering of $M$ and $\varphi : \Omega \to L$ a mapping such that for any $B \in \Omega$, $\varphi(B) \in C_e(B)$. Then $\Omega$ consists of blocks of $\approx_{e^2}$ and $K = \varphi(\Omega)$ satisfies (4.2).

2. If a finite $K \subseteq L$ satisfies (4.2) then $\Omega = \{B_e(b) \mid b \in K\} \subseteq 2^L$ is a set of blocks of $\approx_{e^2}$ that forms a covering of $M$. Moreover $\varphi : \Omega \to L$ defined by $\varphi(B_e(b)) = b$ satisfies $\varphi(B) \in C_e(B)$.

**Proof.** 1. Since $C_e(B) \neq \emptyset$, the first assertion follows from Lemma 2.10. Since $\Omega$ is a covering of $M$, for every $a \in M$ there exists $B \in \Omega$ containing $a$. From the definition of $C_e(B)$, we get $a \leftrightarrow \varphi(B) \geq e$ and from $\varphi(B) \in K$ we get $\bigvee_{b \in K} a \leftrightarrow b \geq e$ proving (4.2).

2. Every $B \in \Omega$ is a closed ball with radius $e$, hence also a block of $\approx_{e^2}$ due to Lemma 2.10. For $a \in M$ let $c \in K$ be such that $a \leftrightarrow c = \bigvee_{b \in K} a \leftrightarrow b$ (such $c$ exists since $K$ is finite and $L$ is linearly ordered). (4.2) implies $a \leftrightarrow c \geq e$, hence $a \in B_e(c)$ and $\Omega$ is a covering of $M$. $\varphi(B) \in C_e(B)$ means $c \in C_e(B_e(c))$ which is true due to 1. of Lemma 2.9.

**Example 4.2.** Let $L = [0, 1]^2$ with Lukasiewicz structure, $M = L$, $e = (0.25, 0.25)$. Then
\[
K = \{(0, 0), (0.5, 0), (1, 0), (0, 0.5), (0, 1)\}
\]
satisfies (4.2) because $\text{appr}(M, K) = (1, 1) \geq e$. However, $\Omega = \{B_e(b) \mid b \in K\}$ is not a covering of $M$ because $(1, 1)$ does not belong to any $B_e(b), b \in K$. Hence, 2. from Theorem 4.1 does not hold for non-linear residuated lattices.

We now present two algorithms which provide solutions to Problem 1. The first algorithm constructs $K$ going through $L$ upwards.

**Algorithm 4.3.**
1: **INPUT:** $M$, $e$
2: **OUTPUT:** $K$ satisfying 1. and 2. of Problem 1
3: $K \leftarrow \emptyset$
4: **while** $M$ is not empty **do**
5: \(\min \leftarrow \min(M)\)
6: \(e \to \min\) to $K$
7: **remove from** $M$ **every element** $\leq (e \otimes e) \to \min$
8: endwhile 
9: return $K$

The second one constructs $K$ going through $L$ downwards.

Algorithm 4.4.

1: INPUT: $M$, $e$
2: OUTPUT: $K$ satisfying 1. and 2. of Problem 1
3: $K ← ∅$
4: while $M$ is not empty do
5: $\max ← \max(M)$
6: add $e \otimes \max$ to $K$
7: remove from $M$ every element $\geq e \otimes e \otimes \max$
8: endwhile
9: return $K$

Lemma 4.5. 1. Let the set $K^n$ produced by Algorithm 4.3 consist of elements $k^n_1 < \cdots < k^n_p$, let $K \subseteq L$, consisting of elements $k_1 < \cdots < k_q$, approximate $M$ to degree at least $e$. Then $q \geq p$ and for any $i$, $1 \leq i \leq p$, $k^n_i \geq k_i$.

2. Let the set $K^l$ produced by Algorithm 4.4 consist of elements $k^l_1 < \cdots < k^l_p$, let $K \subseteq L$, consisting of elements $k_1 < \cdots < k_q$, approximate $M$ to degree at least $e$. Then $q \geq p$ and for any $i$, $1 \leq i \leq p$, $k^l_i \leq k_i$.

Proof. 1. Let $Ω = \{B_e(b) \mid b ∈ K\}$. By part 2 of Theorem 4.1, $Ω$ is a covering of $M$.

Suppose that the set of all $i ≤ \min(p, q)$ such that $k^n_i < k_i$ is not empty and denote its least element by $j$. Since $k^n_j = e \rightarrow \bigwedge M$ then $k^l_j$ is equal to the greatest $a ∈ L$ such that $a \leftrightarrow \bigwedge M \geq e$ (Theorem 2.4), which means that $j > 1$. Denote by $\min$ the least element remaining in $M$ after $j - 1$ steps. From $\min > k^n_{j-1} \geq k_{j-1}$ and $\min \notin B_e(k^n_{j-1})$ we obtain $\min \notin B_e(k_{j-1})$. Since $k^l_j$ is equal to the greatest $b ∈ L$ such that $\min \leftrightarrow b \geq e$ then also $\min \notin B_e(k_j)$. Thus, $Ω$ is not a covering of $M$, which is a contradiction.

It remains to be shown that $q \geq p$. Indeed, if $q \leq p - 1$ then $k_q \leq k^n_{p-1} < \bigvee M$. Since $\bigvee M \notin B(k^n_{p-1})$ then also $\bigvee M \notin B(k_q)$ and $Ω$ is not a covering of $M$.

Part 2. can be proved similarly. 

As the next two theorems show, the algorithms indeed produce solutions to Problem 1.

Theorem 4.6. Algorithms 4.3 and 4.4 are correct. They terminate after at most $O(|M|)$ steps.

Proof. Since $(e \otimes e) \rightarrow \min \geq \min$, at least one element of $M$, namely $\min$, is removed from $M$ in every step in Algorithm 4.3. For Algorithm 4.4, since $e \otimes e \otimes \max \leq \max$, $\max$ is removed from $M$ in every step. As a result, the algorithms terminate after at most $O(|M|)$ steps.

For Algorithm 4.3: In every step, all elements $a ∈ M \cap B$, where $B = [\min, (e \otimes e) \rightarrow \min]$, are removed from $M$ and, at the same time, $e \rightarrow \min$ is added to $K$. By Theorem 2.4, $e \otimes e \rightarrow \min$ is the greatest among all $a ∈ L$ for which $\min \leftrightarrow a \geq e \otimes e$. Hence, $B$ is a block of $≈_2$ and by Lemma 2.10, $C_e(B)$ is non-empty. Set $ϕ(B) = e \rightarrow \min$. By Theorem 2.2, $ϕ(B) ∈ C_e(B)$. Thus, condition 1
of Problem 1 follows from part 1 of Theorem 4.1. Condition 2 follows directly from Lemma 4.5.

The proof for Algorithm 4.4 is similar. \hfill \Box

Furthermore, the algorithms provide upper and lower bounds for every set \( K \) which is a correct output for Problem 1.

**Theorem 4.7.** Let the sets \( K^u \) and \( K^l \) produced by Algorithm 4.3 and Algorithm 4.4 consist of elements \( k_1^u < \cdots < k_p^u \) and \( k_1^l < \cdots < k_p^l \), respectively. If \( K \) consisting of \( k_1 < \cdots < k_p \), satisfies \( \text{appr}(M, K') \geq e \), then

\[
k_1^l \leq k_1 \leq k_1^u, \ldots, k_p^l \leq k_p \leq k_p^u.
\]

**Proof.** Follows directly from Lemma 4.5. \hfill \Box

The following example shows that not every \( K = \{k_1, \ldots, k_p\} \) for which \( k_1^l \leq k_i \leq k_i^u \) satisfies \( \text{appr}(M, K) \geq e \).

**Example 4.8.** Consider the standard Lukasiewicz algebra on \( L = [0,1] \), \( M = \{0.5, 0.7, 0.8\} \), and \( e = 0.9 \). Then \( K^l = \{0.4, 0.7\} \) and \( K^u = \{0.6, 0.9\} \). Let \( K = \{0.4, 0.9\} \). Then \( \text{appr}(M, K) = 0.8 < e \).

Although the set \( K \) produced by Algorithm 4.3 or Algorithm 4.4 is optimal in that it is one of the smallest sets for which \( \text{appr}(M, K) \geq e \), there can be a set \( K' \) of the same size, i.e., \( |K'| = |K| \), for which \( \text{appr}(M, K') > \text{appr}(M, K) \), i.e., \( K' \) provides a better approximation of \( M \) than \( K \). From this point of view, the output set \( K \) from Algorithm 4.3 and Algorithm 4.4 can be improved. Namely, it is easily seen from the description of Algorithm 4.3 and Algorithm 4.4 that the set

\[
\{B_e(k) \cap M \mid k \in K\}
\]

forms a partition of \( M \). In general, \( k \) is not an optimal central point of \( B_e(k) \cap M \). Therefore, we can improve \( K \) by replacing every \( k \in K \) by an optimal central point of \( B_e(k) \cap M \).

By Theorem 3.9, if \( L \) has square roots, then any element from

\[
\left[ \sqrt{\bigcap (B_e(k) \cap M)} \otimes \bigvee (B_e(k) \cap M), \right.

\left. \sqrt{\bigwedge (B_e(k) \cap M)} \rightarrow \bigwedge (B_e(k) \cap M) \right]
\]

can be used to replace \( k \). For instance, for \( M = \{0.5, 0.7, 0.8\} \) and \( K = K^l = \{0.4, 0.7\} \) from Example 4.8, \( B_{0.9}(0.7) \cap M = \{0.7, 0.8\} \) and \( B_{0.9}(0.4) \cap M = \{0.5\} \). Hence, 0.4 can be replaced by 0.5 (optimal central point of \( \{0.5\} \)) and 0.7 can be replaced by 0.75 (optimal central point of \( \{0.7, 0.8\} \)). As a result, we get a set \( \text{opt}(K^l) = \{0.5, 0.75\} \) for which \( \text{appr}(M, \text{opt}(K^l)) = 0.95 > 0.9 = \text{appr}(M, K) \). In addition, we have \( K^u = \{0.6, 0.9\} \), \( \text{opt}(K^u) = \{0.6, 0.8\} \), but this time \( \text{appr}(M, \text{opt}(K^u)) = 0.9 = \text{appr}(M, K) \).

As the next example shows, such improvement does not, in general, satisfy condition 3. of Problem 2. That is, replacement of points \( k \) in \( K \) by better points \( k' \) which cover the same part of \( M \), i.e., for which \( B_e(k) \cap M = B_e(k') \cap M \), does not result in the best approximating set with size \( |K| \).

**Example 4.9.** Consider the standard Lukasiewicz algebra on \( L = [0,1] \), \( M = \{0, 0.1, 0.3, 0.7, 0.9, 1\} \), and \( e = 0.9 \). Then \( K^l = \{0, 0.2, 0.6, 0.9\} \) and \( K^u = \{0.1, 0.4, 0.8, 1\} \), \( \text{opt}(K^l) = \{0, 0.2, 0.7, 0.95\} \), \( \text{opt}(K^u) = \{0.05, 0.3, 0.8, 1\} \), \( \text{appr}(M, K^l) = \) ...
0.9, \text{appr}(M, K^u) = 0.9, \text{appr}(M, \text{opt}(K^u)) = 0.9, \text{appr}(M, \text{opt}(K^v)) = 0.9. However, \(K = \{0.05, 0.3, 0.7, 0.95\}\) is a solution to Problem 2 for which \text{appr}(M, K) = 0.95.

In what follows, we present an algorithm that provides a solution to Problem 2 provided the largest admissible radii of subsets of \(L\) exist and may be determined. Due to Theorem 3.9, an important class of residuated lattices that satisfies this assumption consists of residuated lattices with square roots.

Let thus \(M = \{m_1 \leq \cdots \leq m_n\}\) and denote by \(r(A)\) the largest admissible radius of \(A \subseteq L\).

**Algorithm 4.10.**

1: INPUT: \(M, e\)
2: OUTPUT: \(K\) solving Problem 2
3: \(K' \leftarrow \text{output of Algorithm 4.3 for } M, e\)
4: \(q \leftarrow |K'|\)
5: \(e' \leftarrow e\)
6: repeat
7: \(K \leftarrow K'\)
8: for \(i = 1\) to \(n - 1\) do
9: \(r' \leftarrow \min\{r([m_i, m]) | m \in M, r([m_i, m]) > e\}\)
10: if \(e' < r'\) then \(e' \leftarrow r'\)
11: endfor
12: \(e \leftarrow e'\)
13: \(K' \leftarrow \text{output of Algorithm 4.3 for } M, e\)
14: until \(|K'| > q\)
15: return \(K\)

**Theorem 4.11.** Algorithm 4.10 is correct. It terminates after at most \(O(|M|^4)\) steps.

*Proof.* The algorithm starts by computing a set \(K'\) with \(q\) elements that approximates \(M\) to degree at least \(e\). The aim of loop 6–14 is to compute another set of \(q\) elements that approximates \(M\) to a largest possible degree \(d\), larger than original \(e\). To this end, the loop in 8–11 determines the least candidate \(e'\) for such \(d\). This is easily seen by realizing that \(r([m_i, m])\) is the radius of \([m_i, m]\) for any optimal central point of \([m_i, m]\), see Theorem 3.1. Any such optimal central point covers \([m_i, m]\) and is a potential element of a new set of \(q\) elements that approximates \(M\) to a higher degree than the lastly computed \(K'\). For \(e'\), Algorithm 4.3 in line 13 computes a new set \(K'\) that approximates \(M\) to degree at least \(e'\). Due to Lemma 4.5, this set has more than \(q\) elements if and only if no set with \(q\) elements approximates \(M\) to degree \(e'\) (or larger). In this case, the candidate \(e'\) may not be used a lower approximation of \(d\) and the loop 6–14 terminates. In the other case, \(e'\) is taken as a lower approximation of \(d\) and the loop 6–14 is entered again to produce and test another, larger candidate, until such a candidate exists. The last computed set \(K'\) with \(q\) elements is therefore the output to Problem 2.

Within loop 8–11, \(r([m_i, m])\) is evaluated at most \(\frac{|M|(|M| - 1)}{2} = O(|M|^2)\) times. Loop 8–11 is run within loop 6–14 which itself is run at most \(\frac{|M|(|M| - 1)}{2} = O(|M|^2)\) times, yielding the total of \(O(|M|^4)\) steps in the worst case. \(\square\)
Note that after Algorithm 4.10 finishes, we may run Algorithm 4.4 on $M$ and $e'$ where $e'$ is the value in Algorithm 4.10 for which the output of Algorithm 4.10 was computed. According to Theorem 4.7, the outputs produced by Algorithm 4.10 and Algorithm 4.4 provide us with lower and upper bounds for every solution to Problem 2.

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References