in this way is denoted by $\underline{\mathfrak{B}}(G, M, I)$ and is called the **concept lattice** of the context (G, M, I). ◇

Example 2. The context in Example 1 has 19 concepts. The line diagram in Figure 1.2 represents the concept lattice of this context.

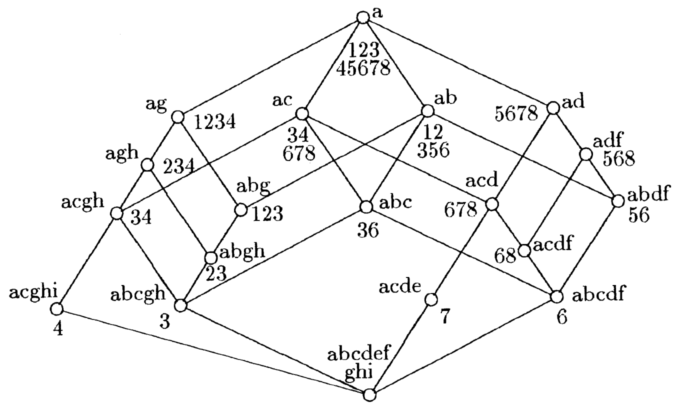

Figure 1.2 Concept lattice for the context of Figure 1.1

Theorem 3 (The Basic Theorem on Concept Lattices). *The concept lattice $\underline{\mathfrak{B}}(G, M, I)$ is a complete lattice in which infimum and supremum are given by:*

$$\bigwedge_{t\in T}(A_t, B_t) = \left(\bigcap_{t\in T} A_t, \left(\bigcup_{t\in T} B_t\right)''\right)$$

$$\bigvee_{t\in T}(A_t, B_t) = \left(\left(\bigcup_{t\in T} A_t\right)'', \bigcap_{t\in T} B_t\right).$$

A complete lattice V is isomorphic to $\underline{\mathfrak{B}}(G, M, I)$ if and only if there are mappings $\tilde{\gamma} : G \to V$ and $\tilde{\mu} : M \to V$ such that $\tilde{\gamma}(G)$ is supremum-dense in V, $\tilde{\mu}(M)$ is infimum-dense in V and gIm is equivalent to $\tilde{\gamma}g \le \tilde{\mu}m$ for all $g \in G$ and all $m \in M$. In particular, $V \cong \underline{\mathfrak{B}}(V, V, \le)$.

Proof of the Basic Theorem. First, we will explain the formula for the infimum. Since $A_t = B_t'$ for each $t \in T$,

$$\left(\bigcap_{t\in T} A_t, \left(\bigcup_{t\in T} B_t\right)''\right)$$

by Proposition 11 can be transformed into

$$\left(\left(\bigcup_{t\in T} B_t\right)', \left(\bigcup_{t\in T} B_t\right)''\right),$$

i.e., it has the form (X', X'') and is therefore certainly a concept. That this can only be the infimum, i.e., the largest common subconcept of the concepts (A_t, B_t), follows immediately from the fact that the extent of this concept is exactly the intersection of the extents of (A_t, B_t). The formula for the supremum is substantiated correspondingly. Thus, we have proven that $\underline{\mathfrak{B}}(G, M, I)$ is a complete lattice.

Now we prove, first for the special case $V = \underline{\mathfrak{B}}(G, M, I)$, the existence of mappings $\tilde{\gamma}$ and $\tilde{\mu}$ with the required properties. We set

$$\tilde{\gamma}g := (\{g\}'', \{g\}') \text{ for } g \in G$$

$$\text{and} \quad \tilde{\mu}m := (\{m\}', \{m\}'') \text{ for } m \in M.$$

As claimed, we have $\tilde{\gamma}g \le \tilde{\mu}m \iff \{g\}'' \subseteq \{m\}' \iff \{g\}' \supseteq \{m\} \iff m \in \{g\}' \iff gIm$. Furthermore, on account of the formulas proved above,

$$\bigvee_{g\in A}(\{g\}'', \{g\}') = (A, B) = \bigwedge_{m\in B}(\{m\}', \{m\}''),$$

holds for every concept (A, B), i.e., $\tilde{\gamma}(G)$ is supremum-dense and $\tilde{\mu}(M)$ is infimum-dense in $\underline{\mathfrak{B}}(G, M, I)$. More generally, if $V \cong \underline{\mathfrak{B}}(G, M, I)$ and $\varphi : \underline{\mathfrak{B}}(G, M, I) \to V$ is an isomorphism, we define $\tilde{\gamma}$ and $\tilde{\mu}$ by

$$\tilde{\gamma}g := \varphi(\{g\}'', \{g\}') \text{ for } g \in G$$

$$\text{and } \tilde{\mu}m := \varphi(\{m\}', \{m\}'') \text{ for } m \in M.$$

The properties claimed for these mappings are proved in a similar fashion.

If, conversely, V is a complete lattice and

$$\tilde{\gamma} : G \to V, \tilde{\mu} : M \to V$$

are mappings with the properties stated above, then we define

$$\varphi : \underline{\mathfrak{B}}(G, M, I) \to V$$

by

$$\varphi(A, B) := \bigvee\{\tilde{\gamma}(g) \mid g \in A\}.$$

Evidently, φ is order-preserving. In order to prove that φ is an isomorphism, we have to demonstrate that φ^{-1} exists and is also order-preserving. Therefore, we define

$$\psi x := (\{g \in G \mid \tilde{\gamma}g \le x\}, \{m \in M \mid x \le \tilde{\mu}m\}),$$

for $x \in V$ and demonstrate that ψx is a concept of (G, M, I):

$$\begin{aligned} h \in \{g \in G \mid \tilde{\gamma} g \leq x\} &\Leftrightarrow \tilde{\gamma} h \leq x \\ &\Leftrightarrow \tilde{\gamma} h \leq \tilde{\mu} n \text{ for all } n \in \{m \in M \mid x \leq \tilde{\mu} m\} \\ &\Leftrightarrow hIn \text{ for all } n \in \{m \in M \mid x \leq \tilde{\mu} m\} \\ &\Leftrightarrow h \in \{m \in M \mid x \leq \tilde{\mu} m\}'. \end{aligned}$$

The second condition follows correspondingly. We have defined a map $\psi : V \to \mathfrak{B}(G, M, I)$, and we can read off directly from the definition that ψ is order-preserving. Now we prove that $\varphi = \psi^{-1}$. We have

$$\varphi \psi x = \bigvee \{\tilde{\gamma} g \mid g \in G, \tilde{\gamma} g \leq x\} = x,$$

since $\tilde{\gamma}(G)$ is supremum-dense in $\boldsymbol{V}$. On the other hand, $\varphi(A, B) = \bigwedge\{\tilde{\mu} m \mid m \in B\}$, since $\tilde{\mu}(M)$ is infimum-dense in $\boldsymbol{V}$, and consequently

$$\begin{aligned} \psi\varphi(A, B) &= \psi \bigwedge \{\tilde{\mu} m \mid m \in B\} \\ &= (\{g \in G \mid \tilde{\gamma} g \leq \bigwedge \{\tilde{\mu} m \mid m \in B\}\}, \{\ldots\}') \\ &= (\{g \in G \mid \tilde{\gamma} g \leq \tilde{\mu} m \text{ for all } m \in B\}, \{\ldots\}') \\ &= (\{g \in G \mid gIm \text{ for all } m \in B\}, \{\ldots\}') \\ &= (B', B'') = (A, B). \end{aligned}$$

If we choose for a complete lattice $\boldsymbol{V}$ specifically $G := V$, $M := V$, $I := \leq$ and $\tilde{\gamma}$ as well as $\tilde{\mu}$ to be the identity of V, we obtain $\boldsymbol{V} \cong \mathfrak{B}(G, M, I)$. □

The Duality Principle for Concept Lattices. Let (G, M, I) be a context. Then (M, G, I^{-1}) is also a context, in fact,

$$\mathfrak{B}(M, G, I^{-1}) \cong \mathfrak{B}(G, M, I)^d,$$

and

$$(B, A) \mapsto (A, B)$$

is an isomorphism.

In other words: if we exchange the roles of objects and attributes, we obtain the dual concept lattice. Thus, the Duality Principle extends to concept lattices.

The mappings $\tilde{\gamma}$ and $\tilde{\mu}$ which appear in the Basic Theorem indicate how the context can be identified in the concept lattice. This is elaborated by the following definition.

Definition 22. For an object $g \in G$ we write g' instead of $\{g\}'$ for the **object intent** $\{m \in M \mid gIm\}$ of the object g. Correspondingly, $m' := \{g \in G \mid gIm\}$ is the **attribute extent** of the attribute m. Retaining the symbols used in the Basic Theorem, we write γg for the **object concept** (g'', g') and μm for the **attribute concept** (m', m''). ◇

The line diagram in Figure 1.2 indicates the intent and the extent of every concept. The labelling can be simplified considerably by putting down each object and each attribute only once, namely at the circle for the respective object and attribute concept (see Figure 1.3). It is still possible to read off the context as well as all extents and intents from the line diagram: If one looks for the extent belonging to one of the little circles which represent the concepts, it consists of the objects located at this circle or the circles which can be reached by descending line paths from this circle. Correspondingly, the intent can be found by following all line paths going upwards from the circle and noting down the attributes assigned to these circles.

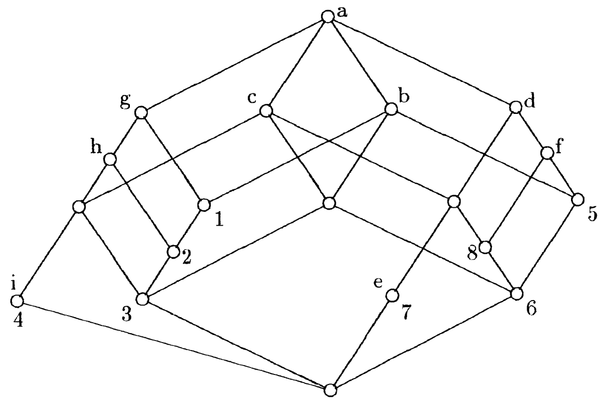

Figure 1.3 Line diagram with reduced labelling.

The sparing, reduced labelling enables us to enter the full names of the objects and attributes of the context in Figure 1.1 into the diagram. This improves the readability of the diagram, as can be seen in Figure 1.4.

1.2 Context and Concept Lattice

A context can be easily reconstructed from the system of all its concepts. G and M appear as the extent and the intent of the trivial *boundary concepts*: The set of all objects is the extent of the largest concept, $(\emptyset', \emptyset'') = (G, G')$. Dually, M is the intent of the least concept, $(\emptyset'', \emptyset') = (M', M)$. The incidence relation I is given by

$$I = \bigcup \{A \times B \mid (A, B) \in \mathfrak{B}(G, M, I)\}.$$

It is even easier to read off the context from the concept lattice, as the Basic Theorem shows. On the other hand, concept lattices of different contexts

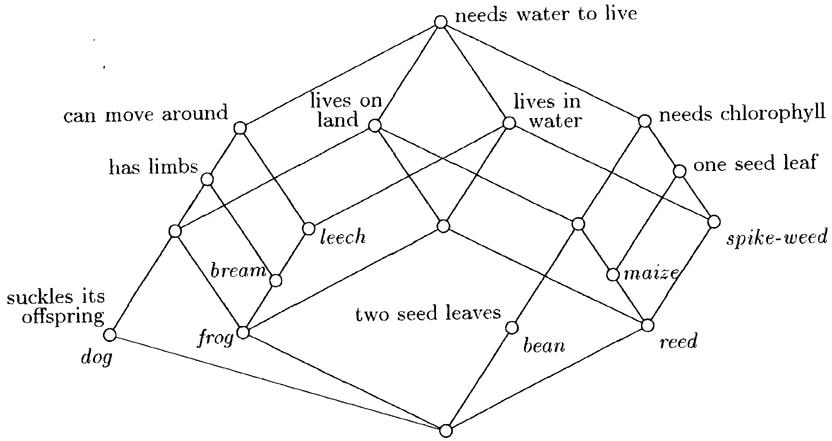

Figure 1.4 Concept lattice for the educational film "Living beings and water".

can well be isomorphic. The context manipulations which do not alter the structure of the concept lattice include the merging of objects with the same intents and attributes with the same extents, respectively:

Definition 23. A context (G, M, I) is called **clarified**, if for any objects $g, h \in G$ from $g' = h'$ it always follows that $g = h$ and, correspondingly, $m' = n'$ implies $m = n$ for all $m, n \in M$. ◇

Example 3. Figure 1.5 shows a context which represents the service offers of an office supplies business. Below the clarified context.

Another feature which has no influence on the structure of the concept lattice are attributes which can be written as a combination of other attributes. More precisely: If $m \in M$ is an attribute and $X \subseteq M$ is a set of attributes with $m \notin X$ but $m' = X'$, then the attribute concept μm is the infimum of the attribute concepts $\mu x, x \in X$, i.e., the set $\mu(M \setminus \{m\})$ is also infimum-dense in $\mathfrak{B}(G, M, I)$, and according to the Basic Theorem

$$\mathfrak{B}(G, M, I) \cong \mathfrak{B}(G, M \setminus \{m\}, I \cap (G \times (M \setminus \{m\}))).$$

The removal of **reducible attributes**, i.e., of attributes with $\bigwedge$-reducible attribute concepts and of **reducible objects**, i.e., of objects with $\bigvee$-reducible object concepts, is called **reducing** the context. **Full rows** and **full columns** are always reducible; thereby we mean objects g with $g' = M$ and attributes m with $m' = G$, respectively.

Definition 24. A clarified context (G, M, I) is called **row reduced**, if every object concept is $\bigvee$-irreducible, and **column reduced**, if every attribute

	Furniture	Computers	Copy-machines	Type-writers	Specialized machines
Consulting	×	×	×	×	×
Planning	×	×			
Assembly and installation	×	×	×	×	×
Instruction		×	×	×	×
Training, workshops		×			
Original spare parts and accessories	×	×	×	×	×
Repairs	×	×	×	×	×
Service contracts		×	×	×	

	Furniture	Computers	Copy machines and typewriters	Specialized machines
Consulting, assembly and installation, original spare parts and accessories, repairs	×	×	×	×
Planning	×	×		
Instruction		×	×	×
Training, workshops		×		
Service contracts		×	×	

Figure 1.5 Context and clarified context.