1.3 Many-valued Contexts

In standard language the word "attribute" is not only used for properties which an object may or may not have. Attributes such as "colour", "weight", "sex", "grade" have *values*. We call them *many-valued attributes*, in contrast to the *one-valued attributes* considered so far.

Definition 27. A **many-valued context** (G, M, W, I) consists of sets G, M and W and a ternary relation I between G, M and W (i.e., $I \subseteq G \times M \times W$) for which it holds that

$$(g, m, w) \in I \text{ and } (g, m, v) \in I \quad \text{always imply} \quad w = v.$$

The elements of G are called **objects**, those of M **(many-valued) attributes** and those of W **attribute values**.

$(g, m, w) \in I$ we read as "the attribute m has the value w" for the object g. (G, M, W, I) is called a n-**valued context**, if W has n elements. The many-valued attributes can be regarded as partial maps from G in W. Therefore, it seems reasonable to write $m(g) = w$ instead of $(g, m, w) \in I$. The **domain** of an attribute m is defined to be

$$\mathrm{dom}(m) := \{g \in G \mid (g, m, w) \in I \text{ for some } w \in W\}.$$

The attribute m is called **complete**, if $\mathrm{dom}(m) = G$. A many-valued context is **complete**, if all its attributes are complete. ◇

Like the **one-valued** contexts treated so far, many-valued contexts can be represented by tables, the rows of which are labelled by the objects and the columns labelled by the attributes. The entry in row g and column m then represents the attribute value $m(g)$. If the attribute m does not have a value for the object g, there will be no entry.[6]

Example 5. The many-valued context represented in the upper part of Figure 1.13 shows a comparison of the different possibilities of arranging the engine and the drive chain of a motorcar (cf. Figure 1.12).

Figure 1.12 Drive concepts for motorcars.[7]

[6] Further information on the role of the "empty cells" in a context will be given in the notes at the end of the chapter.

[7] Source: *Schlag nach! 100 000 Tatsachen aus allen Wissensgebieten*. BI-Verlag Mannheim, 1982.

How can we assign concepts to a many-valued context? We do this in the following way: The many-valued context is transformed into a one-valued one, in accordance with certain rules, which will be explained below. The concepts of this *derived* one-valued context are then interpreted as the concepts of the many-valued context. This interpretation process, however, called **conceptual scaling**, is not at all uniquely determined. The concept system of a many-valued context depends on the scaling. This may at first be confusing, but has proved to be an excellent instrument for a purposeful evaluation of data.

In the process of scaling, first of all each attribute of a many-valued context is interpreted by means of a context. This context is called *conceptual scale*.

Definition 28. A **scale** for the attribute m of a many-valued context is a (one-valued) context $\mathbb{S}_m := (G_m, M_m, I_m)$ with $m(G) \subseteq G_m$. The objects of a scale are called **scale values**, the attributes are called **scale attributes**. ◇

Every context can be used as a scale. Formally there is no difference between a scale and a context. However, we will use the term "scale" only for contexts which have a clear conceptual structure and which bear meaning. Some particularly simple contexts are used as scales time and again. A summary (in tabular form) of the most important ones can be found at the end of the next section.

As already mentioned, the choice of the scale for the attribute m is not mathematically compelling, it is a matter of interpretation. The same is true for the second step in the process of **scaling**, the joining together of the scales to make a one-valued context. In the simplest case, this can be achieved by putting together the individual scales without connecting them. This is described below as *plain scaling*. Particularly when dealing with numerical scales this may well be unsatisfactory. In this case we need the scaling by means of a *composition operator*. For details we refer to the pointers at the end of the chapter.

In the case of **plain scaling** the derived one-valued context is obtained from the many-valued context (G, M, W, I) and the scale contexts $\mathbb{S}_m$, $m \in M$ as follows: The object set G remains unchanged, every many-valued attribute m is replaced by the scale attributes of the scale $\mathbb{S}_m$. If we imagine a many-valued context as represented by a table, we can visualize plain scaling as follows: Every attribute value $m(g)$ is replaced by the row of the scale context $\mathbb{S}_m$ which belongs to $m(g)$. A detailed description will be given in the following definition, for which we first introduce an abbreviation: The attribute set of the derived context is the disjoint union of the attribute sets of the scales involved. In order to make sure that the sets are disjoint, we replace the attribute set of the scale $\mathbb{S}_m$ by

$$\dot{M}_m := \{m\} \times M_m.$$

as in Definition 8 (p. 4).

Definition 29. If (G, M, W, I) is a many-valued context and $\mathbb{S}_m$, $m \in M$ are scale contexts, then the **derived context with respect to plain scaling** is the context (G, N, J) with

$$N := \bigcup_{m \in M} \dot{M}_m,$$

and

$$gJ(m,n) :\Longleftrightarrow m(g) = w \text{ and } wI_m n.$$

$\diamond$

Example 6. We obtain the one-valued context in Figure 1.13 as the derived context of the many-valued context presented above it, if we use the following scales:

$\mathbb{S}_{De} := \mathbb{S}_{Dl} :=$

	++	+	−
++	×	×	
+		×	
−			×

$\mathbb{S}_R :=$

	++	+	− −
++	×	×	
+		×	
− −			×

$\mathbb{S}_S :=$

	u	o	n	u/n
u	×			
o		×		
n			×	
u/n				×

$\mathbb{S}_E := \mathbb{S}_M :=$

	++	+	−	− −
++	×	×		
+		×		
−			×	
− −			×	×

$\mathbb{S}_C :=$

	vl	l	m	h
vl	×	×		
l		×		
m			×	
h				×

If we had used the scale $\mathbb{S}_E$ for the attributes De, Dl and R as well, the derived context would have only turned out slightly different. The concept lattice is shown in Figure 1.14.

The formal definition of a context permits turning relations originating from any domain into contexts and examining their concept lattices, i.e., even contexts where an *interpretation* of the sets G and M as "objects" or "attributes" appears artificial. This is the case with many contexts from mathematics, and in this way we obtain concept lattices which often have structural properties occurring very rarely with empirical data sets. Nevertheless, these contexts are also of great importance for data analysis. They can be used for example as "ideal structures" or as scales for the scaling introduced above. The scales which are used by far most frequently, the *elementary scales* will be introduced now. Other scales will follow in the next section.

We will start with the definition of some operations which permit the construction of new contexts from given ones.

	De	Dl	R	S	E	C	M
Conventional	poor	good	good	understeering	good	medium	excellent
Front-wheel	good	poor	excellent	understeering	excellent	very low	good
Rear-wheel	excellent	excellent	very poor	oversteering	poor	low	very poor
Mid-engine	excellent	excellent	good	neutral	very poor	low	very poor
All-wheel	excellent	excellent	good	understeering/neutral	good	high	poor

	De			Dl			R			S				E				C				M			
	++	+	−	++	+	−	++	+	− −	u	o	n	u/n	++	+	−	− −	vl	l	m	h	++	+	−	− −
Conventional			×		×			×		×					×					×		×	×		
Front-wheel		×				×	×	×		×				×	×			×	×				×		
Rear-wheel	×	×		×	×				×		×					×			×				×		
Mid-engine	×	×		×	×			×				×				×	×		×					×	×
All-wheel	×	×		×	×			×					×		×						×			×	

De := drive efficiency empty; Dl := drive efficiency loaded; R := road holding/handling properties; S := self-steering effect; E := economy of space; C := cost of construction; M := maintainability
++ := excellent; + := good; − := poor; − − := very poor; u := understeering;
o := oversteering; n := neutral; vl := very low; l := low, m := medium; h := high.

Figure 1.13 A many-valued context: Drive concepts for motorcars. Below a derived one-valued context.

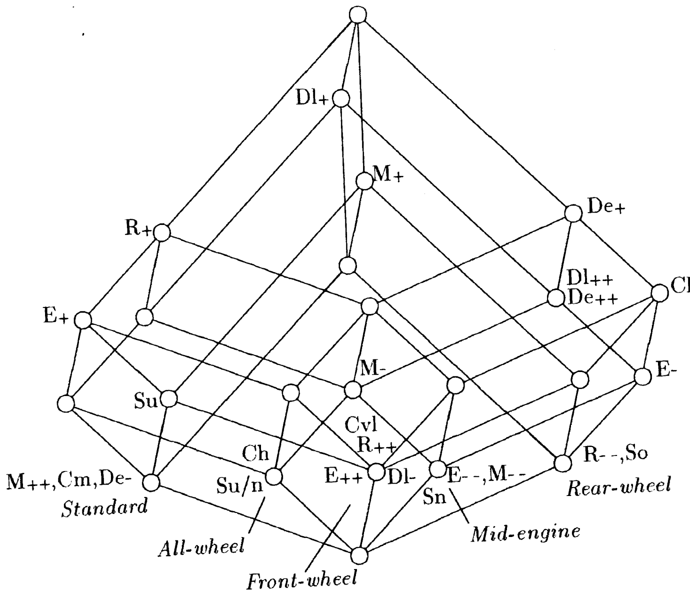


Figure 1.14 Concept lattice for the context of drive concepts.

Definition 30. Let $\mathbb{K} := (G, M, I)$, $\mathbb{K}_1 := (G_1, M_1, I_1)$ and $\mathbb{K}_2 := (G_2, M_2, I_2)$ be contexts. We will use the abbreviations $\dot{G}_j := \{j\} \times G_j$, $\dot{M}_j := \{j\} \times M_j$ and $\dot{I}_j := \{((j,g),(j,m)) \mid (g,m) \in I_j\}$ for $j \in \{1,2\}$ in the following definition. It is:

$\mathbb{K}^c := (G, M, (G \times M) \setminus I)$
the **complementary context** to $\mathbb{K}$,

$\mathbb{K}^d := (M, G, I^{-1})$
the **dual context** to $\mathbb{K}$,

and, if $G = G_1 = G_2$,

$\mathbb{K}_1 \mid \mathbb{K}_2 := (G, \dot{M}_1 \cup \dot{M}_2, \dot{I}_1 \cup \dot{I}_2)$
the **apposition** of $\mathbb{K}_1$ and $\mathbb{K}_2$,

as well as dually, if $M = M_1 = M_2$,

$\dfrac{\mathbb{K}_1}{\mathbb{K}_2} := (\dot{G}_1 \cup \dot{G}_2, M, \dot{I}_1 \cup \dot{I}_2)$
the **subposition** of $\mathbb{K}_1$ and $\mathbb{K}_2$.

$\mathbb{K}_1 \dot{\cup} \mathbb{K}_2 := (\dot{G}_1 \cup \dot{G}_2, \dot{M}_1 \cup \dot{M}_2, \dot{I}_1 \cup \dot{I}_2)$
is the **disjoint union** of $\mathbb{K}_1$ and $\mathbb{K}_2$.

The context $\mathbb{K}^{cd}$ is called the **contrary context** to $\mathbb{K}$. ◇

By using $\dot{G}_i$ for $\{i\} \times G_i$ and $\dot{M}_i$, respectively, we intend to make sure that the sets are disjoint. However, strictly speaking, apposition and subposition under this definition become non-associative. We will overlook this fact and tacitly identify the contexts

$$(\ \mathbb{K}_1 \mid \mathbb{K}_2\ ) \mid \mathbb{K}_3 \quad \text{and} \quad \mathbb{K}_1 \mid (\ \mathbb{K}_2 \mid \mathbb{K}_3\ )\ .$$

The same applies to the subposition, even to hybrid forms of the two operations. We do not distinguish between

$$\frac{\mathbb{K}_1 \mid \mathbb{K}_2}{\mathbb{K}_3 \mid \mathbb{K}_4} \quad \text{and} \quad \frac{\mathbb{K}_1}{\mathbb{K}_3} \,\Bigg|\, \frac{\mathbb{K}_2}{\mathbb{K}_4}\ .$$

The two abbreviations

$$\times := (G, M, G \times M)$$
$$\varnothing := (G, M, \emptyset)$$

are occasionally used without further describing the sets G and M, if they are evident from the context. For example

$$\begin{array}{c|c} \mathbb{K}_1 & \times \\ \hline \varnothing & \mathbb{K}_2 \end{array}$$

denotes the context $(\dot{G}_1 \cup \dot{G}_2, \dot{M}_1 \cup \dot{M}_2, \dot{I}_1 \cup \dot{I}_2 \cup (\dot{G}_1 \times \dot{M}_2))$, the concept lattice of which is isomorphic to the **vertical sum** of the concept lattices $\mathfrak{B}(\mathbb{K}_1)$ and $\mathfrak{B}(\mathbb{K}_2)$ (provided that $\mathbb{K}_1$ does not contain a full column and $\mathbb{K}_2$ does not contain a full row, cf. 4.3).

Each extent of $\mathbb{K}_1 \dot{\cup} \mathbb{K}_2$, apart from the extent $\dot{G}_1 \cup \dot{G}_2$, is entirely contained in one of the sets $\dot{G}_i$. The corresponding applies to the intents. Therefore, the concept lattice $\boldsymbol{V} := \mathfrak{B}(\mathbb{K}_1 \dot{\cup} \mathbb{K}_2)$ is a **horizontal sum,** i.e., it is the union $\boldsymbol{V} = \boldsymbol{V}_1 \dot{\cup} \boldsymbol{V}_2$ of two sublattices which only overlap in the smallest and the largest element: $\boldsymbol{V}_1 \cap \boldsymbol{V}_2 = \{0_V, 1_V\}$. Provided that there are no full rows or columns in $\mathbb{K}_1$ and $\mathbb{K}_2$, we have $\boldsymbol{V}_i \cong \mathfrak{B}(\mathbb{K}_i)$ or, more generally, $\boldsymbol{V}_i = \mathfrak{B}(\dot{G}_1 \cup \dot{G}_2, \dot{M}_1 \cup \dot{M}_2, \dot{I}_i)$.

In Definition 28 we postulated that the values of the many-valued attribute had to be the objects of the scale. In the following *standardized scale* we frequently use $\mathbf{n} := \{1, 2, \ldots, n\}$ as the object set. In this case, in order to scale a many-valued attribute, we first have to rename the objects. The appropriate definitions for the isomorphy of scales will be introduced later, in Chapter 7.3 (p. 258 ff.).

Definition 31 (elementary scales, see also Figure 1.15)

Nominal scales. $\mathbb{N}_n := (\mathbf{n}, \mathbf{n}, =)$.

Nominal scales are used to scale attributes, the values of which mutually exclude each other. If an attribute for example has the values {*masculine, feminine, neuter*}, the use of a nominal scale suggests itself. We thereby obtain a *partition* of the objects into extents. In this case, the classes correspond to the values of the attribute.

	1	2	3	4
1	×			
2		×		
3			×	
4				×

The Nominal Scale $\mathbb{N}_4$.

(One-dimensional) ordinal scales. $\mathbb{O}_n := (\mathbf{n}, \mathbf{n}, \leq)$.

$\mathbb{O}_4 =$

	1	2	3	4
1	×	×	×	×
2		×	×	×
3			×	×
4				×

Ordinal scales scale many-valued attributes, the values of which are ordered and where each value implies the weaker ones. If an attribute has for instance the values {*loud, very loud, extremely loud*} ordinal scaling suggests itself. The attribute values then result in a chain of extents, interpreted as a hierarchy.

(One-dimensional) interordinal scales. $\mathbb{I}_n := (\mathbf{n}, \mathbf{n}, \leq) \mid (\mathbf{n}, \mathbf{n}, \geq)$.

$\mathbb{I}_4 =$

	≤ 1	≤ 2	≤ 3	≤ 4	≥ 1	≥ 2	≥ 3	≥ 4
1	×	×	×	×	×			
2		×	×	×	×	×		
3			×	×	×	×	×	
4				×	×	×	×	×

Questionnaires often offer opposite pairs as possible answers, as for example *active–passive, talkative–taciturn* etc., allowing a choice of intermediate values. In this case, we have a *bipolar* ordering of the values. This kind of attributes lend themselves to scaling by means of an interordinal scale. The extents of the interordinal scale are precisely the intervals of values, in this way, the betweenness relation is reflected conceptually. However, bipolar attributes often also lend themselves to *biordinal* scaling:

Biordinal scales. $\mathbb{M}_{n,m} := (\mathbf{n}, \mathbf{n}, \leq) \,\dot{\cup}\, (\mathbf{m}, \mathbf{m}, \geq)$.

$\mathbb{M}_{4,2} =$

	≤ 1	≤ 2	≤ 3	≤ 4	≥ 5	≥ 6
1	×	×	×	×		
2		×	×	×		
3			×	×		
4				×		
5					×	
6					×	×

In common usage we often use opposite pairs not in the sense of an interordinal scale, but simpler: each object is assigned one of the two poles, allowing graduations. The values {*very low, low, loud, very loud*} for example suggest this way of scaling: *loud* and *low* mutually exclude each other, *very loud* implies *loud*, *very low* implies *low*. We also find this kind of *partition with a hierarchy* in the names of the school marks: An excellent performance obviously is also very good, good, and satisfactory, but not unsatisfactory or a fail.

The **dichotomic scale.** $\mathbb{D} := (\{0,1\}, \{0,1\}, =)$

The dichotomic scale constitutes a special case, since it is isomorphic to the scales $\mathbb{N}_2$ and $\mathbb{M}_{1,1}$ and closely related to $\mathbb{I}_2$. It is frequently used to scale attributes with values of the kind {*yes, no*}.

	0	1
0	×	
1		×

A special case of plain scaling which frequently occurs is the case that all many-valued attributes can be interpreted with respect to the same scale or family of scales. Thus we speak of a **nominally scaled context**, if all scales $\mathbb{S}_m$ are nominal scales etc. We call a many-valued context **nominal**, if the nature of the data suggests nominal scaling; a many-valued context is called an **ordinal context** if for each attribute the set of values is ordered in a natural way. An example is presented in Figure 1.16, see also Figure 1.17.

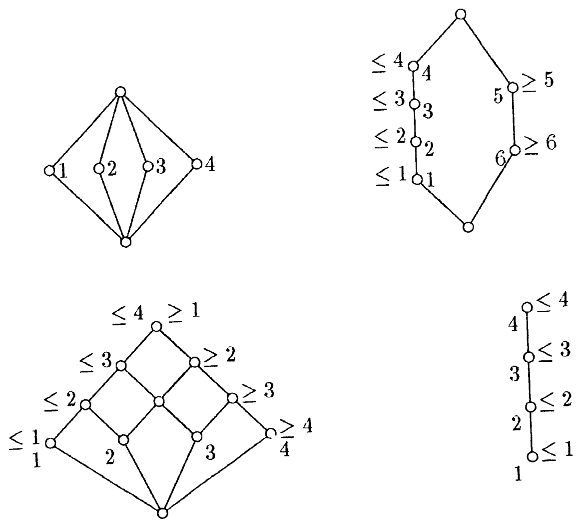


Figure 1.15 The concept lattices of the elementary scales are named after the scales. The figure shows a **nominal lattice**, $\underline{\mathfrak{B}}(\mathbb{N}_4)$, a **biordinal lattice**, $\underline{\mathfrak{B}}(\mathbb{M}_{4,2})$, an **interordinal lattice**, $\underline{\mathfrak{B}}(\mathbb{I}_4)$, and an **ordinal lattice**, $\underline{\mathfrak{B}}(\mathbb{O}_4)$. The ordinal lattice $\underline{\mathfrak{B}}(\mathbb{O}_n)$ is isomorphic to the n-element **chain** C_n.

	Forum Romanum	B	GB	M	P
1	Arch of Septimus Severus	*	*	**	*
2	Arch of Titus	*	**	**	
3	Basilica Julia			*	
4	Basilica of Maxentius	*			
5	Phocas column		*	**	
6	Curia				*
7	House of the Vestals			*	
8	Portico of Twelve Gods		*	*	*
9	Tempel of Antonius and Fausta	*	*	* * *	*
10	Temple of Castor and Pollux	*	**	* * *	*
11	Temple of Romulus		*		
12	Temple of Saturn			**	*
13	Temple of Vespasian			**	
14	Temple of Vesta		**	**	*

Figure 1.16 Example of an ordinal context: Ratings of monuments on the Forum Romanum in different travel guides (B = Baedecker, GB = Les Guides Bleus, M = Michelin, P = Polyglott). The context becomes ordinal through the number of stars awarded. If no star has been awarded, this is rated zero.

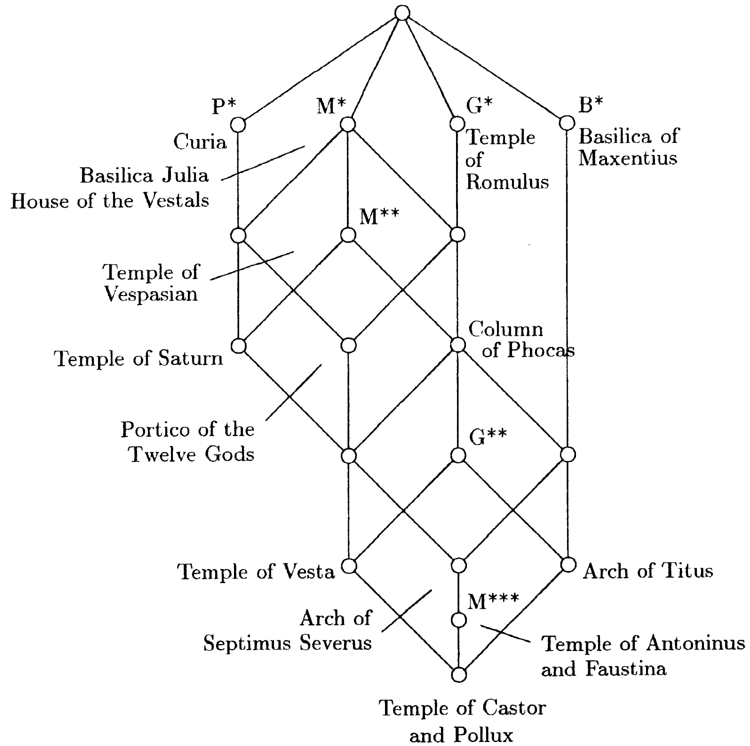


Figure 1.17 The concept lattice of the ordinal context from Figure 1.16.

1.4 Context Constructions and Standard Scales

We have formulated the following frequently used sum and product constructions for two contexts each, but the definitions can be easily generalized to any number of contexts. The additional statements on the concept lattices of the resulting contexts carry over.

Definition 32. The **direct sum** of two contexts is defined by[8]

$$\mathbb{K}_1 + \mathbb{K}_2 := (\dot{G}_1 \cup \dot{G}_2, \dot{M}_1 \cup \dot{M}_2, \dot{I}_1 \cup \dot{I}_2 \cup (\dot{G}_1 \times \dot{M}_2) \cup (\dot{G}_2 \times \dot{M}_1))$$

◇

The concept lattice of a sum of contexts is isomorphic to the product of its concept lattices. In the case of two contexts we therefore obtain

$$\underline{\mathfrak{B}}(\mathbb{K}_1 + \mathbb{K}_2) \cong \underline{\mathfrak{B}}(\mathbb{K}_1) \times \underline{\mathfrak{B}}(\mathbb{K}_2),$$

since (A, B) is a concept of $\mathbb{K}_1 + \mathbb{K}_2$ if and only if $(A \cap \dot{G}_i, B \cap \dot{M}_i)$ is a concept of $\dot{\mathbb{K}}_i := (\dot{G}_i, \dot{M}_i, \dot{I}_i)$, for $i \in \{1, 2\}$. This means that the isomorphism is given by $(A, B) \mapsto ((A \cap \dot{G}_1, B \cap \dot{M}_1), (A \cap \dot{G}_2, B \cap \dot{M}_2))$.

Definition 33. The **semiproduct** is defined by

$$\mathbb{K}_1 \mathbin{\bar{\underline{\times}}} \mathbb{K}_2 := (G_1 \times G_2, \dot{M}_1 \cup \dot{M}_2, \nabla)$$

with

$$(g_1, g_2)\nabla(j, m) :\iff g_j I_j m \qquad \text{for } j \in \{1, 2\}.$$

◇

The extents of the semiproduct are precisely the sets of the form $A_1 \times A_2$, each set A_j being an extent of $\mathbb{K}_j$. This also yields the structure of the concept lattice $\underline{\mathfrak{B}}(\mathbb{K}_1 \mathbin{\bar{\underline{\times}}} \mathbb{K}_2)$: Essentially, the concept lattice is the product of the concept lattices of the factor contexts, though there is a modification regarding the zero elements. Precisely, the instruction for the construction reads as follows: Provided that the extent of the corresponding concept is empty, we remove the zero element from each of the extents $\underline{\mathfrak{B}}(\mathbb{K}_j)$. Then we form the product of these ordered sets and, if we have previously removed an element, we add a new zero element to make a complete lattice. This lattice is then isomorphic to the concept lattice of the semiproduct.

Definition 34. The **direct product** is given by

$$\mathbb{K}_1 \times \mathbb{K}_2 := (G_1 \times G_2, M_1 \times M_2, \nabla)$$

$$\text{with } (g_1, g_2)\nabla(m_1, m_2) :\iff g_1 I_1 m_1 \text{ or } g_2 I_2 m_2.$$

◇

[8] For the notation see Definition 8 (p. 4). A more general definition is given in Section 5.1.

The concept lattice of the direct product is called the *tensor product* of the concept lattices of the factor contexts. We will later discuss the tensor product in more detail (Sections 4.4, 5.4). We obtain the cross table of the direct product by replacing each empty cell in the table of $\mathbb{K}_1$ by a copy of $\mathbb{K}_2$ and each cross by a square full of crosses of the size of $\mathbb{K}_2$. For an example see Figures 4.19 (page 164) and 4.20.

Another context construction, the so-called **substitution sum**, where a context is inserted into an other context, will be described in section 4.3. The sum and the product of reduced contexts are reduced (cf. Corollary 74, p. 166). Reducible objects or attributes with empty intents or extents may occur in the case of the disjoint union. Semi products of reduced contexts are reduced if the factors (allowing for one exception at most) are **atomistic**, i.e., if they satisfy $g' \subseteq h' \Rightarrow g = h$.

It is easy to state numerous simple arithmetical rules for context constructions, which are useful for some proofs. In particular, the direct product is (up to isomorphism) commutative and associative; it is distributive over the direct sum, the apposition and the subposition. We note down one of these results for later:

Proposition 16.

$$(\mathbb{K}_1 + \mathbb{K}_2) \times \mathbb{K}_3 = (\mathbb{K}_1 \times \mathbb{K}_3) + (\mathbb{K}_2 \times \mathbb{K}_3).$$

Proof. We may assume that the three contexts $\mathbb{K}_i =: (G_i, M_i, I_i)$, $i \in \{1, 2, 3\}$, have disjoint object sets and disjoint attribute sets. By

$$(G_1 \cup G_2) \times G_3 = (G_1 \times G_3) \cup (G_2 \times G_3)$$

and

$$(M_1 \cup M_2) \times M_3 = (M_1 \times M_3) \cup (M_2 \times M_3),$$

the two contexts of the proposition have the same objects and attributes. For the incidence we find the same on both sides as well, namely

$$(g, h) I (m, n) \iff \begin{cases} g \in G_1 \text{ and } m \in M_2 & \text{or} \\ g \in G_2 \text{ and } m \in M_1 & \text{or} \\ h I_3 n & \text{or} \\ g \in G_1, m \in M_1 \text{ and } g I_1 m & \text{or} \\ g \in G_2, m \in M_2 \text{ and } g I_2 m. & \end{cases}$$

□

We now state a list of interesting context families. Many of them have proved to be useful as scales. We provide a summary of these scales, including their basic meanings, in Figure 1.26 at the end of this section. Besides, these contexts serve as a reservoir of examples for mathematical reasoning.

(1) For every set S the **contranominal scale**

$$\mathbb{N}^c_S := (S, S, \neq)$$

is reduced. The concepts of this context are precisely the pairs $(A, S\setminus A)$ for $A \subseteq S$. The concept lattice is isomorphic to the power-set lattice of S, and thus has $2^{|S|}$ elements. If $S = \{1, 2, \ldots, n\}$ we write $\mathbb{N}^c_n$.

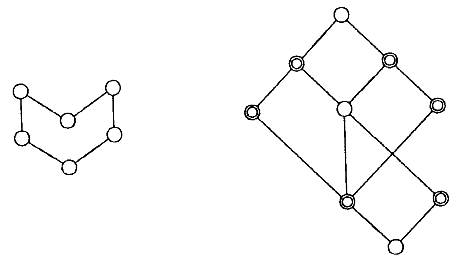


Figure 1.18 Example of an ordered set $(P, \leq)$ and its completion $\underline{\mathfrak{B}}(P, P, \leq)$.

(2) From an arbitrary ordered set $\mathbf{P} := (P, \leq)$ we obtain the **general ordinal scale**

$$\mathbb{O}_{\mathbf{P}} := (P, P, \leq).$$

Its concepts are precisely the pairs (X, Y) with $X, Y \subseteq P$ where X is the set of all lower bounds of Y and Y is the set of all upper bounds of X. This concept lattice is called the **Dedekind-MacNeille completion** of the ordered set $\mathbf{P}$. It is the smallest complete lattice in which $\mathbf{P}$ can be order-embedded, in the sense of the following theorem:

Theorem 4. (Dedekind's Completion Theorem) *For an ordered set* $(P, \leq)$

$$\iota x := ((x], [x)) \quad \textit{for } x \in P$$

defines an embedding ι *of* $(P, \leq)$ *in* $\underline{\mathfrak{B}}(P, P, \leq)$*; moreover,* $\iota \bigvee X = \bigvee \iota X$ *or* $\iota \bigwedge X = \bigwedge \iota X$ *if the supremum or infimum of* X*, respectively, exists in* $(P, \leq)$*. If* κ *is an arbitrary embedding of* $(P, \leq)$ *in a complete lattice* V*, then there is always also an embedding* λ *of the ordered set* $\underline{\mathfrak{B}}(P, P, \leq)$ *in* V *with* $\kappa = \lambda \circ \iota$.

Proof. Evidently, the concepts of $(P, P, \leq)$ are precisely the pairs (A, B) with $A, B \subseteq P$ and

$$\begin{aligned} A = B^{\downarrow} &:= \{x \in P \mid x \leq y \text{ for all } y \in B\}, \\ B = A^{\uparrow} &:= \{y \in P \mid x \leq y \text{ for all } x \in A\}; \end{aligned}$$

in particular, all pairs $((x], [x))$ with $x \in P$ are concepts of $(P, P, \leq)$, which confirms ι as an embedding. If the supremum of X exists in $(P, \leq)$, then

$$\left[\bigvee X\right) = \bigcap_{x \in X} [x),$$

i.e., $\iota \bigvee X =$

$$= \left(\left(\bigvee X\right], \left[\bigvee X\right)\right) = \left(\left(\bigcap_{x\in X}[x)\right)^{\downarrow}, \bigcap_{x\in X}[x)\right) = \bigvee((x], [x)) = \bigvee \iota X.$$

The equation for existing infima is shown dually.

With respect to the missing part of the proof we refer to Proposition 33 (p. 99). □

(3) From an arbitrary ordered set $\mathbf{P} := (P, \leq)$ we furthermore obtain the reduced context

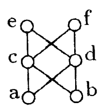
$$\mathbb{O}^{cd}_{\mathbf{P}} := (P, P, \not\geq),$$

which is called the **contraordinal scale.** In this case, the concepts are precisely the pairs (X, Y) with the following properties:
- $X \cup Y = P$ and $X \cap Y = \emptyset$,
- X is an **order ideal** in $\mathbf{P}$, i.e., $x \in X$ and $z \leq x$ always imply $z \in X$. Because of $X \cup Y = P$ and $X \cap Y = \emptyset$ this is equivalent to:
- Y is an **order filter** in $\mathbf{P}$, i.e., $y \in Y$ and $y \leq z$ always imply $z \in Y$.

The context $(P, P, \not\geq)$ is doubly founded, since

$$x \swarrow y \iff x \nearrow y \iff x = y$$

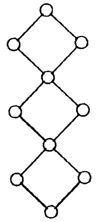
holds for $x, y \in P$. Hence if x is an object and y is an attribute with $x \not I y$ (i.e., $x \geq y$), then $x \nearrow x$ and $x' = P \setminus (x] \supset P \setminus (y] = y'$, hold for the *attribute* x, as required by Definition 26.
The concept lattice $\underline{\mathfrak{B}}(P, P, \not\geq)$ is isomorphic to the lattice of the order ideals of $\mathbf{P}$. A look at (1) shows that all concepts of the contraordinal scale are concepts of the contranominal scale $\mathbb{N}^c_P$ as well. We will prove later (Theorem 13, p. 112) that for this reason $\underline{\mathfrak{B}}(P, P, \not\geq)$ is a complete sublattice of $\underline{\mathfrak{B}}(P, P, \neq)$, which means that these lattices are completely distributive. Birkhoff's theorem (Theorem 39, p. 220) shows that the lattices constructed in this way, are precisely the doubly founded completely distributive lattices. In particular, every finite distributive lattice is isomorphic to the concept lattice of a contraordinal scale. The dual lattice, i.e., $\underline{\mathfrak{B}}(P, P, \not\leq)$, is often denoted by $2^{\mathbf{P}}$, because it is also isomorphic to the lattice of the order-preserving maps of $\mathbf{P}$ to the two-element lattice.



$(P,\leq)$

	a	b	c	d	e	f
a		×	×	×	×	×
b	×		×	×	×	×
c				×	×	×
d			×		×	×
e						×
f					×	

$(P,P,\not\geq)$



$\mathfrak{B}(P,P,\not\geq)$

Figure 1.19 An ordered set $(P,\leq)$, the corresponding contraordinal scale and its concept lattice, i.e., the ideal lattice of $(P,\leq)$.

(4) We obtain an interesting special case of (3) by choosing the power-set of a set S as our ordered set $\boldsymbol{P}$, i.e., by considering the context

$$(\mathfrak{P}(S),\mathfrak{P}(S),\not\supseteq).$$

Because of $A \not\supseteq B \iff B\cap(S\setminus A)\neq\emptyset$, this context is isomorphic to

$$(\mathfrak{P}(S),\mathfrak{P}(S),\Delta) \quad \text{with} \quad X\Delta Y :\iff (X\cap Y)\neq\emptyset.$$

The concept lattice is called the **free completely distributive lattice** FCD(S). If for $S:=\{1,2,\ldots,n\}$ we denote the context $(\mathfrak{P}(S),\mathfrak{P}(S),\not\supseteq)$ by $\mathbb{A}_n$, we can state an easy recursion rule for the generation of these contexts:

$$\mathbb{A}_0 = \boxed{\emptyset} \quad \text{and} \quad \mathbb{A}_{n+1} = \begin{array}{c|c} \mathbb{A}_n & \times \\ \hline \mathbb{A}_n & \mathbb{A}_n \end{array}.$$

The construction can be generalized by taking an ordered set $(S,\leq)$ as the base set, the set $\mathcal{OI}(S,\leq)$ of the order ideals of $(S,\leq)$ as the object set and the set $\mathcal{OF}(S,\leq)$ of the order filters of $(S,\leq)$ as the attribute set. The concept lattice

$$\mathrm{FCD}(S,\leq) := (\mathcal{OI}(S,\leq),\mathcal{OF}(S,\leq),\Delta)$$

is called the **free completely distributive lattice over the ordered set** $(S,\leq)$.

(5) For an arbitrary ordered set $(P,\leq)$, we define a **filter** to be a subset of P which is an order filter and in which furthermore any two elements have a common lower bound. Hence $F\subseteq P$ is a filter if and only if the following two conditions are satisfied:
1. From $x\in F$ and $y\geq x$ it follows that $y\in F$,
2. for any two elements $x,y\in F$ there is an $u\in F$ with $u\leq x$ and $u\leq y$.

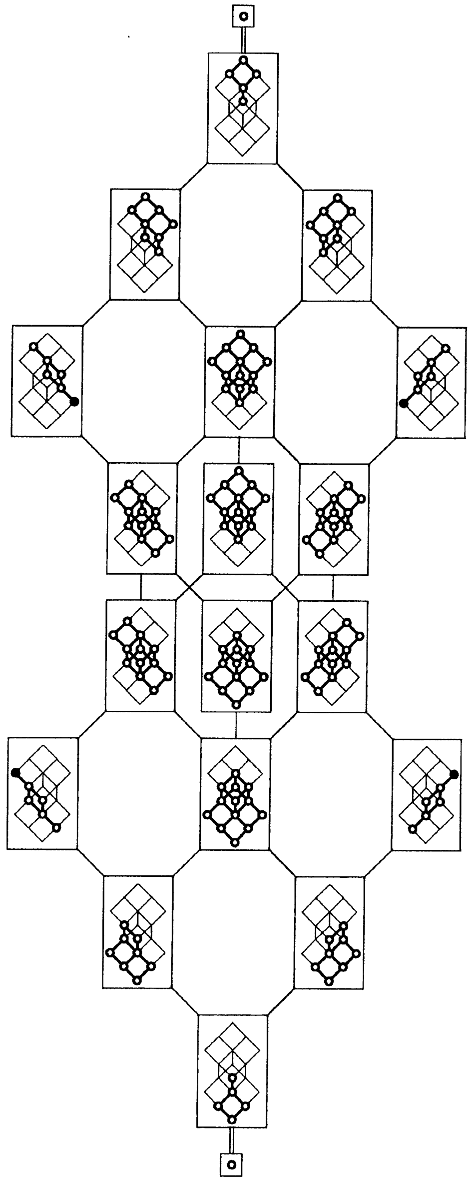


Figure 1.20 A nested line diagram of the free distributive lattice FCD(4). Such diagrams are introduced in 2.2. The one shown here is due to S. Thiele [175]. The method that led to it is explained in Example 14 (p. 215).

Dually, an **ideal** is defined to be a subset of P which is an order ideal and contains a common upper bound for any two elements contained in it. Filters in this sense are among other things the principal filters. Dually, each principal ideal is an ideal. The set of all filters is denoted by $\mathcal{F}(P,\leq)$, the set of all ideals by $\mathcal{I}(P,\leq)$. We obtain the doubly founded context

$$\mathbb{F}_{(P,\leq)} := (\mathcal{F}(P,\leq), \mathcal{I}(P,\leq), \Delta),$$

where again

$$F \Delta I :\Longleftrightarrow F \cap I \neq \emptyset.$$

(6) Again from an ordered set $\boldsymbol{P} := (P,\leq)$ we obtain the **general interordinal scale**

$$\mathbb{I}_{\boldsymbol{P}} := (P,P,\leq) \mid (P,P,\geq),$$

the concept system of which we explain by means of the extents: the attribute extents are precisely the principal ideals and the principal filters of $\boldsymbol{P}$, the object extents are all intersections of those sets. These include all intervals[9]. In general, these are all sets which constitute *intersections of intervals.*

(7) By analogy with (6) we obtain the **convex-ordinal scale**

$$\mathbb{C}_{\boldsymbol{P}} := (P,P,\not\geq) \mid (P,P,\not\leq).$$

In this case, the extents are precisely the convex subsets of $\boldsymbol{P}$, i.e., those subsets which contain with any two elements a and b all elements c with $a \leq c \leq b$.

	1,a	1,b	1,c	1,d	1,e	1,f	2,a	2,b	2,c	2,d	2,e	2,f
a		×	×	×	×	×		×				
b	×		×	×	×	×	×					
c				×	×	×	×	×		×		
d			×		×	×	×	×	×			
e						×	×	×	×	×		×
f					×		×	×	×	×	×	

Figure 1.21 The convex-ordinal scale of the ordered set from Figure 1.19.

(8) Let S be a set and $s \in S$ an arbitrary element. If we now choose G to be the set of all two-element subsets of S and M to be the set of all subsets of $S \setminus \{s\}$, by the definition

$$\{x,y\} \diamond X :\Leftrightarrow |\{x,y\} \cap X| \neq 1$$

[9] in the sense of Definition 5 (p. 3), i.e., only the "closed" intervals.

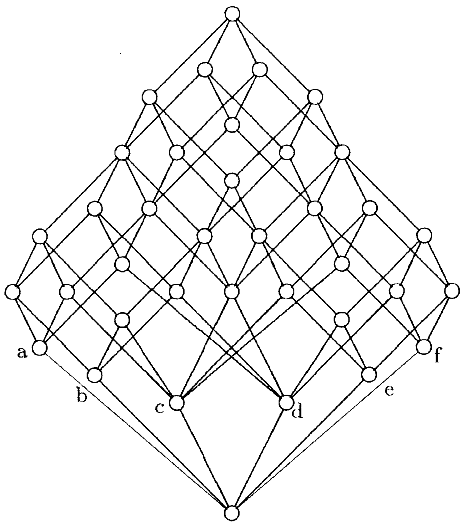


Figure 1.22 The concept lattice of the convex-ordinal scale from Figure 1.21.

we obtain a context $(G,M,\diamond)$ with $\binom{|S|}{2}$ objects and $2^{|S|-1}$ attributes, which is reduced except for one full column. Every extent of this context is a set of two-element subsets of S, i.e., it can be understood as a symmetric reflexive relation on S; actually, the relations occurring are precisely the equivalence relations on S. Hence the concept lattice $\mathfrak{B}(G,M,\diamond)$ is isomorphic to the lattice $\mathfrak{E}(S)$ of equivalence relations. We can give a mnemonic rule for this context series as well. We get $\mathbb{P}_1 := (\emptyset, \{*\}, \emptyset)$ and obtain the $n+1$-st context of this series, $\mathbb{P}_{n+1}$, from the n-th as follows: We form the apposition of $\mathbb{P}_n$ with the cross table $\mathbb{P}_n^{\mathrm{rev}}$, which is identical to $\mathbb{P}_n$, apart from the fact that the columns are written down in the reversed order.

$\mathbb{P}_n$	$\mathbb{P}_n^{\mathrm{rev}}$
$2^n-1 \quad \dots \quad 2^{n-1}$	$2^{n-1}-1 \quad \dots \quad 0$

We add n further rows, which we fill with crosses such that the columns of this subcontext look like the binary representations of the numbers $2^n-1,\dots,0$. An example is given in Figure 1.23.

(9) If R is a symmetric relation on S (easily visualized by the edges of an undirected graph) then with

10011001
32100123
76543210
$\longrightarrow$

×			×	×			×
×	×					×	×
×		×			×		×
×	×	×	×				
×	×			×	×		
×		×		×		×	

Figure 1.23 Context $\mathbb{P}_4$ for the lattice of equivalence relations on a 4-element set.

$$(S, S, R)$$

we obtain a context, the concepts of which are precisely the pairs (A, B), $A \subseteq S$, $B \subseteq S$, which are maximal with respect to the property that each element of A is in the relation R with each element of B (in the visualization these are maximal complete bipartite edge sets). Thus, together with (A, B), (B, A) is also a concept, and the map

$$(A, B) \mapsto (B, A)$$

is a **polarity**, i.e., an order-reversing bijection which is inverse to itself (another term for this is *involutory antiautomorphism*). Conversely, every complete **polarity lattice** (i.e., every complete lattice with a polarity) is isomorphic to the concept lattice of a context (S, S, R) with a symmetric relation R.

If the relation R is irreflexive, the extent and the intent of each concept must be disjoint and we have

$$(A, B) \wedge (B, A) = (\varnothing, \varnothing')$$

$$\text{and } (A, B) \vee (B, A) = (\varnothing', \varnothing),$$

i.e., (A, B) and (B, A) are **complementary** to each other: Their infimum is the smallest, their supremum the largest element of the concept lattice. A lattice with this kind of polarity is called an **ortholattice**; the complete ortholattices are (up to isomorphism) precisely the concept lattices of contexts with an irreflexive, symmetric relation.

There are many examples of such contexts in this book. They can be easily recognized if the cross table is represented symmetric to the main diagonal. The context $\mathbb{K}_{(2,3)}$ in Figure 1.24 is the context of a polarity lattice but not of an ortholattice. The same applies to the context in Figure 5.9 (p. 205), although this only becomes clear after an adroit reassembly of the cross table.

(10) If V is a finite dimensional vector space and V^* is the dual space of V, then

$$(V, V^*, \perp) \quad \text{with} \quad a \perp \varphi :\Longleftrightarrow \varphi a = 0$$

is a doubly founded context, the extents of which are precisely the subspaces of V.

For the special case of the vector spaces over $GF(2)$ there is again a simple recursion for the generation of these contexts: For

$$\mathbb{K}_{(d,2)} := (GF(2)^d, (GF(2)^d)^*, \perp)$$

it is easy to prove that

$$\mathbb{K}_{(d+1,2)} = \frac{\mathbb{K}_{(d,2)} \mid \mathbb{K}_{(d,2)}}{\mathbb{K}_{(d,2)} \mid \mathbb{K}^c_{(d,2)}}.$$

An example is given in Figure 1.24.

×	×	×	×	×	×	×	×
×		×		×		×	
×	×			×	×		
×			×	×			×
×	×	×	×				
×		×			×		×
×	×					×	×
×			×		×	×	

Figure 1.24 $\mathbb{K}_{(3,2)}$, a context derived from the 3-dimensional vector space over the two-element field.

(11) If H is a Hilbert space and $\perp$ is the orthogonality relation, then the concept lattice of the context

$$(H, H, \perp)$$

is isomorphic to the (orthomodular) lattice of the closed subspaces of H; since $(U, U^\perp)$ is a concept for each such subspace U.

(12) The set of all permutations of the set $\{1, \ldots, n\}$ can be given a lattice order in a natural way. For this purpose we call a pair $(\varphi i, \varphi j)$ an **inversion** of the permutation φ if $i < j$ but $\varphi i > \varphi j$. If we order the permutations by

$$\sigma \leq \tau :\Longleftrightarrow \text{ every inversion of } \sigma \text{ is also an inversion of } \tau,$$

we obtain, as proved by Yanagimoto and Okamoto [217], a lattice Σ_n. There is a simple recursion rule for the description of the context: Putting

$$\mathbb{K}_0 := \mathbb{L}_0 := \boxed{\times} \quad \text{and}$$

$$\mathbb{L}_{n+1} := \frac{\varnothing \mid \mathbb{L}_n}{\mathbb{L}_n \mid \mathbb{L}_n}, \qquad \mathbb{K}_{n+1} := \frac{\mathbb{K}_n \mid \mathbb{K}_n}{\mathbb{K}_n \mid \mathbb{L}_n},$$

then we obtain

$$\Sigma_n \cong \mathfrak{B}(\mathbb{K}_n).$$

The contexts $\mathbb{K}_n$ are reduced except for the full rows and full columns. Σ_4 is presented in Figure 1.25.

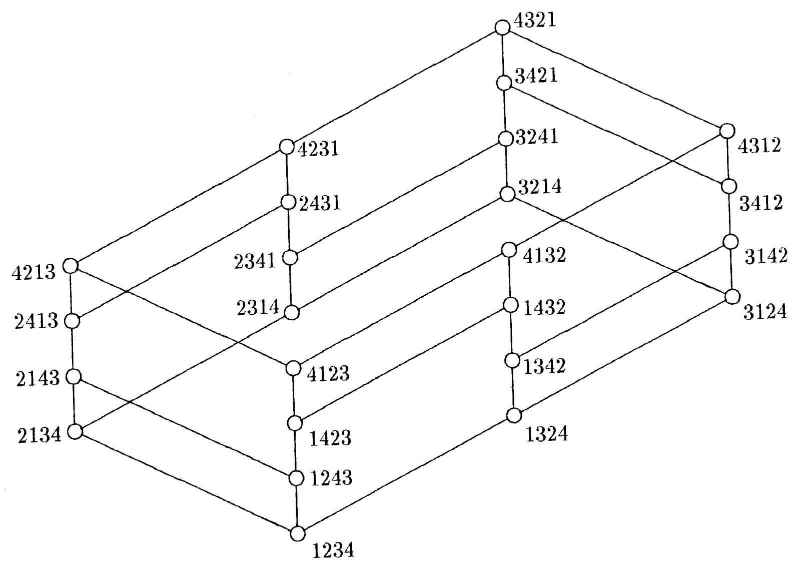


Figure 1.25 The lattice Σ_4 of the permutations of $\{1, 2, 3, 4\}$.

If the ordered sets occurring in the definitions for the standard scales are compounded, for example as a cardinal sum or as a direct product, it is to be expected that the respective scales can be split up. This is true, even if in different ways, as exemplified by the following rules:

Proposition 17.

$$\begin{aligned}
\mathbb{O}_{\mathbf{P}_1+\mathbf{P}_2} &= \mathbb{O}_{\mathbf{P}_1} \dot\cup \mathbb{O}_{\mathbf{P}_2}\\
\mathbb{I}_{\mathbf{P}_1+\mathbf{P}_2} &= \mathbb{I}_{\mathbf{P}_1} \dot\cup \mathbb{I}_{\mathbf{P}_2}\\
\mathbb{O}^{cd}_{\mathbf{P}_1+\mathbf{P}_2} &= \mathbb{O}^{cd}_{\mathbf{P}_1} + \mathbb{O}^{cd}_{\mathbf{P}_2}\\
\mathbb{C}_{\mathbf{P}_1+\mathbf{P}_2} &= \mathbb{C}_{\mathbf{P}_1} + \mathbb{C}_{\mathbf{P}_2}\\
\mathbb{O}^{cd}_{\mathbf{P}_1\times\mathbf{P}_2} &= \mathbb{O}^{cd}_{\mathbf{P}_1} \times \mathbb{O}^{cd}_{\mathbf{P}_2}\\
\mathbb{C}_{\mathbf{P}_1\times\mathbf{P}_2} &= \mathbb{O}^{cd}_{\mathbf{P}_1} \times \mathbb{O}^{cd}_{\mathbf{P}_2} \mid \mathbb{O}^{c}_{\mathbf{P}_1} \times \mathbb{O}^{c}_{\mathbf{P}_2}
\end{aligned}$$

Symbol	Definition	Name	Basic meaning
$\mathbb{O}_\mathbf{P}$	$(P, P, \leq)$	general ordinal scale	hierarchy
$\mathbb{O}_n$	$(\mathbf{n}, \mathbf{n}, \leq)$	one-dimensional ordinal scale	rank order
$\mathbb{N}_n$	$(\mathbf{n}, \mathbf{n}, =)$	nominal scale	partition
$\mathbb{M}_{n_1,\ldots,n_k}$	$\mathbb{O}_{\mathbf{n}_1+\cdots+\mathbf{n}_k}$	multiordinal scale	partition with rank orders
$\mathbb{M}_{m,n}$	$\mathbb{O}_{\mathbf{m}+\mathbf{n}}$	biordinal scale	two-class rank orders
$\mathbb{B}_n$	$(\mathfrak{P}(\mathbf{n}), \mathfrak{P}(\mathbf{n}), \subseteq)$	n-dimensional Boolean scale	dependency of attributes
$\mathbb{G}_{n_1,\ldots,n_k}$	$\mathbb{O}_{\mathbf{n}_1} \boxtimes \cdots \boxtimes \mathbb{O}_{\mathbf{n}_k}$	k-dimensional grid scale	multiple ordering
$\mathbb{O}^{cd}_\mathbf{P}$	$(P, P, \not\geq)$	contraordinal scale	hierarchy and independence
$\mathbb{N}^c_n$	$(\mathbf{n}, \mathbf{n}, \neq)$	contranominal scale	partition and independence
$\mathbb{D}$	$(\{0,1\}, \{0,1\}, =)$	dichotomic scale	dichotomy
$\mathbb{D}_k$	$\underbrace{\mathbb{D} \boxtimes \cdots \boxtimes \mathbb{D}}_{k\text{-times}}$	k-dimensional dichotomic scale	multiple dichotomy
$\mathbb{I}_\mathbf{P}$	$\mathbb{O}_\mathbf{P} \mid \mathbb{O}^d_\mathbf{P}$	general interordinal scale	betweenness relation
$\mathbb{I}_n$	$\mathbb{O}_n \mid \mathbb{O}^d_n$	one-dimensional interordinal scale	linear betweenness relation
$\mathbb{C}_\mathbf{P}$	$\mathbb{O}^{cd}_\mathbf{P} \mid \mathbb{O}^c_\mathbf{P}$	convex-ordinal scale	convex ordering

Figure 1.26 Standardized scales of ordinal type.