TABLE 11.4 GENERALIZED HYPOTHETICAL SYLLOGISMS

Name	Standard intersection	Algebraic product	Bounded difference	Drastic intersection
Early Zadeh $\mathfrak{I}_m$	N	N	N	N
Gaines-Rescher $\mathfrak{I}_s$	Y	Y	Y	Y
Gödel $\mathfrak{I}_g$	Y	Y	Y	Y
Goguen $\mathfrak{I}_\Delta$	N	Y	Y	Y
Kleene-Dienes $\mathfrak{I}_b$	N	N	Y	Y
Lukasiewicz $\mathfrak{I}_a$	N	N	Y	Y
Reichenbach $\mathfrak{I}_r$	N	N	N	N
Willmott $\mathfrak{I}_{wi}$	N	N	N	N
Wu $\mathfrak{I}_{wu}$	Y	Y	Y	Y
$\mathfrak{I}_{ss}$	Y	Y	Y	Y
$\mathfrak{I}_{sg}$	Y	Y	Y	Y
$\mathfrak{I}_{gg}$	Y	Y	Y	Y
$\mathfrak{I}_{gs}$	Y	Y	Y	Y

11.4 MULTICONDITIONAL APPROXIMATE REASONING

The general schema of multiconditional approximate reasoning has the form:

$$\begin{array}{ll} \textit{Rule 1}: & \text{If } \mathcal{X} \text{ is } A_1, \text{ then } \mathcal{Y} \text{ is } B_1 \\ \textit{Rule 2}: & \text{If } \mathcal{X} \text{ is } A_2, \text{ then } \mathcal{Y} \text{ is } B_2 \\ \multicolumn{2}{l}{\dotfill} \\ \textit{Rule } n: & \text{If } \mathcal{X} \text{ is } A_n, \text{ then } \mathcal{Y} \text{ is } B_n \\ \textit{Fact}: & \mathcal{X} \text{ is } A' \\ \hline \textit{Conclusion}: & \mathcal{Y} \text{ is } B' \end{array} \tag{11.16}$$

Given n *if-then* rules, rules 1 through n, and a fact "$\mathcal{X}$ is A'," we conclude that "$\mathcal{Y}$ is B'," where $A', A_j \in \mathcal{F}(X)$, $B', B_j \in \mathcal{F}(Y)$ for all $j \in \mathbb{N}_n$, and X, Y are sets of values of variables $\mathcal{X}$ and $\mathcal{Y}$. This kind of reasoning is typical in fuzzy logic controllers (Chapter 12).

The most common way to determine B' in (11.16) is referred to as a *method of interpolation*. It consists of the following two steps:

Step 1. Calculate the degree of consistency, $r_j(A')$, between the given fact and the antecedent of each *if-then* rule j in terms of the height of intersection of the associated sets A' and A_j. That is, for each $j \in \mathbb{N}_n$,

$$r_j(A') = h(A' \cap A_j)$$

or, using the standard fuzzy intersection,

$$r_j(A') = \sup_{x \in X} \min[A'(x), A_j(x)]. \tag{11.17}$$

Step 2. Calculate the conclusion B' by truncating each set B_j by the value of $r_j(A')$, which expresses the degree to which the antecedent A_j is compatible with the given fact A', and taking the union of the truncated sets. That is,

$$B'(y) = \sup_{j \in \mathbb{N}_n} \min[r_j(A'), B_j(y)] \tag{11.18}$$

for all $y \in Y$.

An illustration of the method of interpolation for two *if-then* rules is given in Fig. 11.3, which is self-explanatory.

The interpolation method is actually a special case of the compositional rule of inference. To show this, assume that R is a fuzzy relation on $X \times Y$ defined by

$$R(x, y) = \sup_{j \in \mathbb{N}_n} \min[A_j(x), B_j(y)] \tag{11.19}$$

for all $x \in X, y \in Y$. Then, B' obtained by (11.18) is equal to $A' \circ R$, where $\circ$ denotes the sup-min composition. This equality can be easily demonstrated. Using (11.18) and (11.17), the following holds:

$$\begin{aligned}
B'(y) &= \sup_{j \in \mathbb{N}_n} \min[r_j(A'), B_j(y)] \\
&= \sup_{j \in \mathbb{N}_n} \min[\sup_{x \in X} \min(A'(x), A_j(x)), B_j(y)] \\
&= \sup_{j \in \mathbb{N}_n} \sup_{x \in X} [\min(A'(x), A_j(x), B_j(y))] \\
&= \sup_{x \in X} \sup_{j \in \mathbb{N}_n} \min[A'(x), \min(A_j(x), B_j(y))] \\
&= \sup_{x \in X} \min[A'(x), \sup_{j \in \mathbb{N}_n} \min(A_j(x), B_j(y))] \\
&= \sup_{x \in X} \min[A'(x), R(x, y)] \\
&= (A' \circ R)(y).
\end{aligned}$$

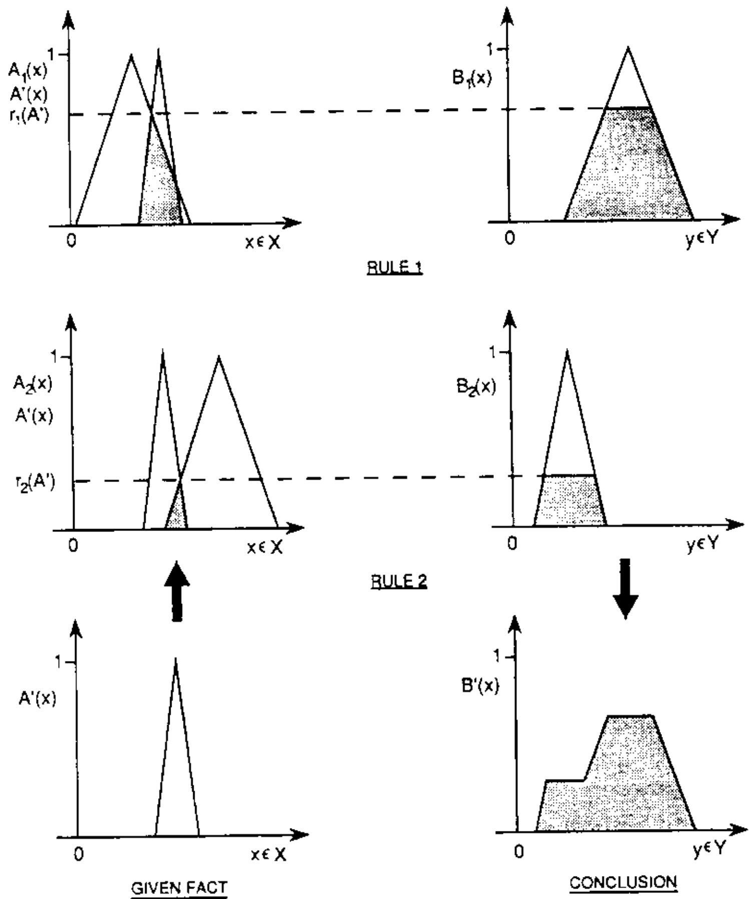

Figure 11.3 Illustration of the method of interpolation.

Hence, $B' = A' \circ R$.

Observe that the fuzzy relation R employed in the reasoning is obtained from the given *if-then* rules in (11.16) in the following way. For each rule j in (11.16), we determine a relation R_j by the formula

$$R_j(x, y) = \min[A_j(x), B_j(y)] \tag{11.20}$$

for all $x \in X, y \in Y$. Then, R is defined by the union of relations R_j for all rules in (11.16). That is,

$$R = \bigcup_{j \in \mathbb{N}_n} R_j. \tag{11.21}$$

In this case, we treat the *if-then* rules as *disjunctive*. This means that we obtain a conclusion for a given fact A' whenever $r_j(A') > 0$ for at least one rule j. When $r_j(A') > 0$, we say that rule j *fires* for the given fact A'.

The *if-then* rules in (11.16) may also be treated as *conjunctive*. In this case, we define

R by the intersection

$$R = \bigcap_{j \in \mathbb{N}_n} R_j. \tag{11.22}$$

We obtain a conclusion for a given fact A' only if $r_j(A') > 0$ for all $j \in \mathbb{N}_n$. That is, to obtain a conclusion, all rules in (11.16) must fire.

The interpretation of the rules in (11.16) as either disjunctive or conjunctive depends on their intended use and the way R_j is obtained. For either interpretation, there are two possible ways of applying the compositional rule of inference: the compositional rule is applied to the fuzzy relation R, after it is calculated by either (11.21) or (11.22); or the compositional rule is applied locally to each relation R_j, and then, the resulting fuzzy sets are combined in either disjunctive or conjunctive ways. Hence, we have the following four possible ways of calculating the conclusion B':

$$B'_1 = A' \circ (\bigcup_{j \in \mathbb{N}_n} R_j), \tag{11.23}$$

$$B'_2 = A' \circ (\bigcap_{j \in \mathbb{N}_n} R_j), \tag{11.24}$$

$$B'_3 = \bigcup_{j \in \mathbb{N}_n} A' \circ R_j, \tag{11.25}$$

$$B'_4 = \bigcap_{j \in \mathbb{N}_n} A' \circ R_j. \tag{11.26}$$

The four distinct fuzzy sets obtained by these formulas are ordered in the way stated in the following theorem.

Theorem 11.6. $B'_2 \subseteq B'_4 \subseteq B'_1 = B'_3$.

Proof: First, we prove that $B'_2 \subseteq B'_4$. For all $y \in Y$,

$$\begin{aligned}
B'_4(y) &= \inf_{j \in \mathbb{N}_n} (A' \circ R_j)(y) \\
&= \inf_{j \in \mathbb{N}_n} \sup_{x \in X} \min[A'(x), R_j(x, y)] \\
&\geq \sup_{x \in X} \inf_{j \in \mathbb{N}_n} \min[A'(x), R_j(x, y)] \\
&= \sup_{x \in X} \min[A'(x), \inf_{j \in \mathbb{N}_n} R_j(x, y)] \\
&= \sup_{x \in X} \min[A'(x), (\bigcap_{j \in \mathbb{N}_n} R_j)(x, y)] \\
&= [A' \circ (\bigcap_{j \in \mathbb{N}_n} R_j)](y) \\
&= B'_2(y).
\end{aligned}$$

Hence, $B'_2 \subseteq B'_4$. Next, we prove that $B'_4 \subseteq B'_1$. This is rather trivial, since

$$A' \circ R_j \subseteq A' \circ (\bigcup_{j \in \mathbb{N}_n} R_j)$$

for all $j \in \mathbb{N}_n$ and, hence,

$$B'_4 = \bigcap_{j \in \mathbb{N}_n} A' \circ R_j \subseteq A' \circ (\bigcup_{j \in \mathbb{N}_n} R_j) = B'_1.$$

Finally, we prove that $B'_1 = B'_3$. For all $y \in Y$,

$$\begin{aligned} B'_1(y) &= \sup_{x \in X} \min[A'(x), \bigcup_{j \in \mathbb{N}_n} R_j(x, y)] \\ &= \sup_{x \in X} \sup_{j \in \mathbb{N}_n} \min[A'(x), R_j(x, y)] \\ &= \sup_{j \in \mathbb{N}_n} \sup_{x \in X} \min[A'(x), R_j(x, y)] \\ &= (\bigcup_{j \in \mathbb{N}_n} A' \circ R_j)(y) \\ &= B'_3(y). \end{aligned}$$

Hence, $B'_1 = B'_3$, which completes the proof. ∎

Let us mention that this theorem is not restricted to the sup-min composition. It holds for any sup-i composition, provided that the t-norm i is continuous.

In general, R_j may be determined by a suitable fuzzy implication, as discussed in Sec. 11.3. That is,

$$R_j(x, y) = \mathcal{J}[A_j(x), B_j(y)] \quad (11.27)$$

is a general counterpart of (11.20). Furthermore, R may be determined by solving appropriate fuzzy relation equations, as discussed in the next section, rather than by aggregating relations R_j.

11.5 THE ROLE OF FUZZY RELATION EQUATIONS

As previously explained, any conditional (*if-then*) fuzzy proposition can be expressed in terms of a fuzzy relation R between the two variables involved. One of the key issues in approximate reasoning is to determine this relation for each given proposition. Once it is determined, we can apply the compositional rule of inference to facilitate our reasoning process.

One way of determining R, which is discussed in Sec. 11.3, is to determine a suitable fuzzy implication $\mathcal{J}$, which operates on fuzzy sets involved in the given proposition, and to express R in terms of $\mathcal{J}$ (see, e.g., (11.12)). As criteria for determining suitable fuzzy implications, we require that the various generalized rules of inference coincide with their classical counterparts. For each rule of inference, this requirement is expressed by a fuzzy relation equation that fuzzy implications suitable for the rule must satisfy. However, the problem of determining R for a given conditional fuzzy proposition may be detached from fuzzy implications and viewed solely as a problem of solving the fuzzy relation equation for R.

As explained in Sec. 11.3, the equation to be solved for modus ponens has the form

$$B = A \overset{i}{\circ} R, \quad (11.28)$$

12

FUZZY SYSTEMS

12.1 GENERAL DISCUSSION

In general, a *fuzzy system* is any system whose variables (or, at least, some of them) range over states that are fuzzy sets. For each variable, the fuzzy sets are defined on some relevant universal set, which is often an interval of real numbers. In this special but important case, the fuzzy sets are fuzzy numbers, and the associated variables are linguistic variables (Sec. 4.2).

Representing states of variables by fuzzy sets is a way of quantizing the variables. Due to the finite resolution of any measuring instrument, appropriate quantization, whose coarseness reflects the limited measurement resolution, is inevitable whenever a variable represents a real-world attribute. For example, when measurements of values of a variable can be obtained only to an accuracy of one decimal digit, two decimal digits, and so on, a particular quantization takes place. The interval of real numbers that represents the range of values of the variable is partitioned into appropriate subintervals. Distinct values within each subinterval are indistinguishable by the measuring instrument involved and, consequently, are considered equivalent. The subintervals are labelled by appropriate real numbers (i.e., relevant real numbers with one significant digit, two significant digits, etc.), and these labels are viewed as *states* of the variable. That is, states of any quantized variable are representatives of equivalence classes of actual values of the variable. Each given state of a quantized variable is associated with uncertainty regarding the actual value of the variable. This uncertainty can be measured by the size of the equivalence class, as explained in Sec. 9.2.

To illustrate the usual quantization just described, let us consider a variable whose range is $[0, 1]$. Assume that the measuring instrument employed allows us to measure the variable to an accuracy of one decimal digit. That is, states of the variable are associated with intervals $[0, .05), [.05, .15), [.15, .25), \dots, [.85, .95), [.95, 1]$ that are labelled, respectively, by their representatives $0, .1, .2, \dots, .9, 1$. This example of quantization is shown in Fig. 12.1a.

Measurement uncertainty, expressed for each measuring instrument by a particular coarseness of states of the associated variable, is an example of *forced uncertainty*. In general, forced uncertainty is a result of information deficiency. Measurement uncertainty,

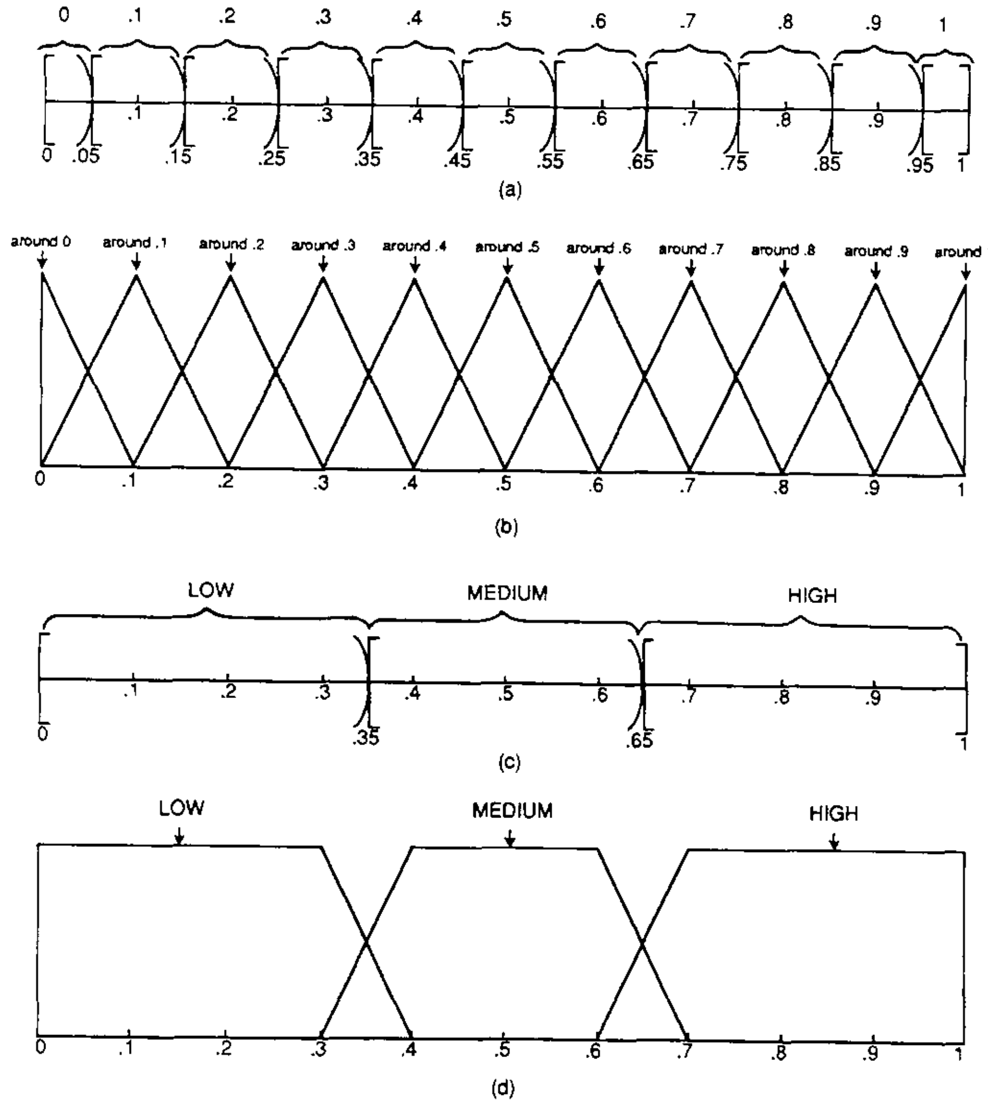

Figure 12.1 Examples of distinct types of quantization: (a) crisp forced; (b) fuzzy forced; (c) crisp opted; (d) fuzzy opted.

for example, results from the principal inability of any measuring instrument to overcome its limiting finite resolution.

Although the usual quantization of variables is capable of capturing limited resolutions of measuring instruments, it completely ignores the issue of measurement errors. While representing states of a variable by appropriate equivalence classes of its values is mathematically convenient, the ever-present measurement errors make this representation highly unrealistic. It can be made more realistic by expressing the states as fuzzy sets. This is illustrated for our previous example in Fig. 12.1b. Fuzzy sets are, in this example, fuzzy numbers with the shown triangular membership functions, and their representations are the linguistic labels *around* 0, *around* .1, *around* .2, and so forth. Fuzzy quantization is often called *granulation.*

Forced uncertainty must be distinguished from *opted uncertainty.* The latter is not a result of any information deficiency but, instead, results from the lack of need for higher certainty. Opted uncertainty is obtained, for example, by quantizing a variable beyond the coarseness induced by the measuring instrument involved. This additional quantization allows us to reduce information regarding the variable to a level desirable for a given task. Hence, while forced uncertainty is a subject of epistemology, opted uncertainty is of a pragmatic nature.

Considering our previous example, assume that we need to distinguish only three states of the variable instead of the eleven states that are made available by our measuring instrument. It is reasonable to label these states as *low, medium*, and *high.* A crisp definition of these states and its more meaningful fuzzy counterpart are shown in Figs. 12.1c and d, respectively.

One reason for eliminating unnecessary information in complex systems with many variables is to reduce the complexity when using the system for a given task. For example, to describe a procedure for parking a car in terms of a set of relevant variables (position of the car relative to other objects on the scene, direction of its movement, speed, etc.), it would not be practical to specify values of these variables with high precision. As is well known, a description of this procedure in approximate linguistic terms is quite efficient. This important role of uncertainty in reducing complexity is well characterized by Zadeh [1973]:

> Given the deeply entrenched tradition of scientific thinking which equates the understanding of a phenomenon with the ability to analyze it in quantitative terms, one is certain to strike a dissonant note by questioning the growing tendency to analyze the behavior of humanistic systems as if they were mechanistic systems governed by difference, differential, or integral equations.
>
> Essentially, our contention is that the conventional quantitative techniques of system analysis are intrinsically unsuited for dealing with humanistic systems or, for that matter, any system whose complexity is comparable to that of humanistic systems. The basis for this contention rests on what might be called the *principle of incompatibility*. Stated informally, the essence of this principle is that as the complexity of a system increases, our ability to make precise and yet significant statements about its behavior diminishes until a threshold is reached beyond which precision and significance (or relevance) become almost mutually exclusive characteristics. It is in this sense that precise analyses of the behavior of humanistic systems are not likely to have much relevance to the real-world societal, political, economic, and other types of problems which involve humans either as individuals or in groups.
>
> An alternative approach... is based on the premise that the key elements in human thinking are not numbers, but labels of fuzzy sets, that is, classes of objects in which the transition from membership to non-membership is gradual rather than abrupt. Indeed, the pervasiveness of fuzziness in human thought processes suggests that much of the logic behind human reasoning is not the traditional two-valued or even multivalued logic, but a logic with fuzzy truths, fuzzy connectives, and fuzzy rules of inference. In our view, it is this fuzzy, and as yet not well-understood, logic that plays a basic role in what may well be one of the most important facets of human thinking, namely, the ability to *summarize* information—to extract from the collection of masses of data impinging upon the human brain those and only those subcollections which are relevant to the performance of the task at hand.

> By its nature, a summary is an approximation to what it summarizes. For many purposes, a very approximate characterization of a collection of data is sufficient because most of the basic tasks performed by humans do not require a high degree of precision in their execution. The human brain takes advantage of this tolerance for imprecision by encoding the "task-relevant" (or "decision-relevant") information into labels of fuzzy sets which bear an approximate relation to the primary data. In this way, the stream of information reaching the brain via the visual, auditory, tactile, and other senses is eventually reduced to the trickle that is needed to perform a specific task with a minimal degree of precision. Thus, the ability to manipulate fuzzy sets and the consequent summarizing capability constitute one of the most important assets of the human mind as well as a fundamental characteristic that distinguishes human intelligence from the type of machine intelligence that is embodied in present-day digital computers.
>
> Viewed in this perspective, the traditional techniques of system analysis are not well suited for dealing with humanistic systems because they fail to come to grips with the reality of the fuzziness of human thinking and behavior. Thus to deal with such systems radically, we need approaches which do not make a fetish of precision, rigor, and mathematical formalism, and which employ instead a methodological framework which is tolerant of imprecision and partial truths.

A lot of work has already been done to explore the utility of fuzzy set theory in various subareas of systems analysis. However, the subject of systems analysis is too extensive to be covered here in a comprehensive fashion. Hence, we can cover only a few representative aspects and rely primarily on the Notes at the end of this chapter to overview the rapidly growing literature on this subject.

The most successful application area of fuzzy systems has undoubtedly been the area of fuzzy control. It is thus appropriate to cover fuzzy control in greater detail than other topics. Our presentation of fuzzy control includes a discussion of the connection between fuzzy controllers and neural networks, the importance of which has increasingly been recognized. Furthermore, we also discuss the issue of fuzzifying neural networks.

12.2 FUZZY CONTROLLERS: AN OVERVIEW

In general, fuzzy controllers are special expert systems (Sec. 11.1). Each employs a knowledge base, expressed in terms of relevant fuzzy inference rules, and an appropriate inference engine to solve a given control problem. Fuzzy controllers vary substantially according to the nature of the control problems they are supposed to solve. Control problems range from complex tasks, typical in robotics, which require a multitude of coordinated actions, to simple goals, such as maintaining a prescribed state of a single variable. Since specialized books on fuzzy controllers are now available (Note 12.2), we restrict our exposition to relatively simple control problems.

Fuzzy controllers, contrary to classical controllers, are capable of utilizing knowledge elicited from human operators. This is crucial in control problems for which it is difficult or even impossible to construct precise mathematical models, or for which the acquired models are difficult or expensive to use. These difficulties may result from inherent nonlinearities, the time-varying nature of the processes to be controlled, large unpredictable environmental disturbances, degrading sensors or other difficulties in obtaining precise and reliable measurements, and a host of other factors. It has been observed that experienced human operators are generally able to perform well under these circumstances.

The knowledge of an experienced human operator may be used as an alternative to a precise model of the controlled process. While this knowledge is also difficult to express in precise terms, an imprecise linguistic description of the manner of control can usually be articulated by the operator with relative ease. This linguistic description consists of a set of control rules that make use of fuzzy propositions. A typical form of these rules is exemplified by the rule

IF the temperature is very high
AND the pressure is slightly low
THEN the heat change should be slightly negative,

where temperature and pressure are the observed state variables of the process, and heat change is the action to be taken by the controller. The vague terms *very high, slightly low,* and *slightly negative* can be conveniently represented by fuzzy sets defined on the universes of discourse of temperature values, pressure values, and heat change values, respectively. This type of linguistic rule has formed the basis for the design of a great variety of fuzzy controllers described in the literature.

A general fuzzy controller consists of four modules: a *fuzzy rule base*, a *fuzzy inference engine*, and *fuzzification/defuzzification modules*. The interconnections among these modules and the controlled process are shown in Fig. 12.2.

A fuzzy controller operates by repeating a cycle of the following four steps. First, measurements are taken of all variables that represent relevant conditions of the controlled process. Next, these measurements are converted into appropriate fuzzy sets to express measurement uncertainties. This step is called a *fuzzification*. The fuzzified measurements are then used by the inference engine to evaluate the control rules stored in the fuzzy rule base. The result of this evaluation is a fuzzy set (or several fuzzy sets) defined on the universe

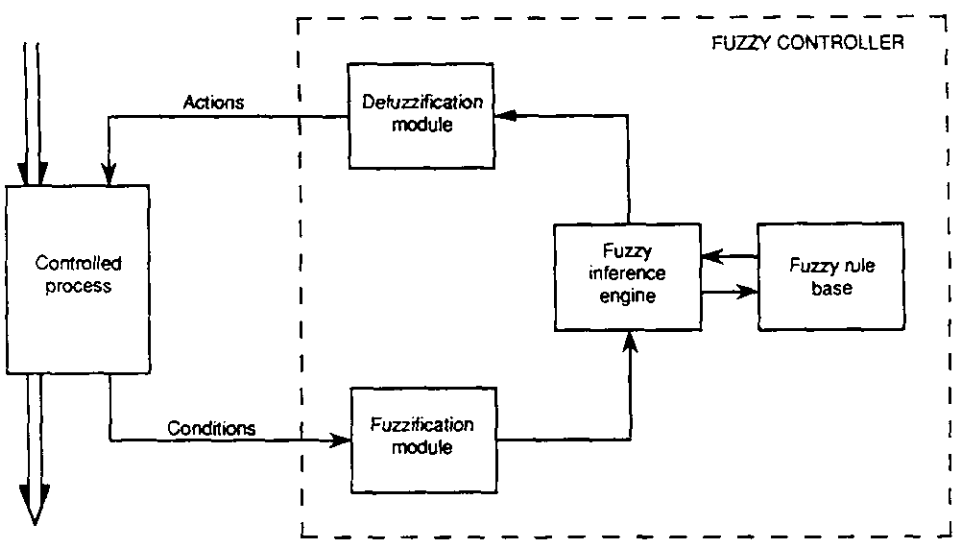

Figure 12.2 A general scheme of a fuzzy controller.

of possible actions. This fuzzy set is then converted, in the final step of the cycle, into a single (crisp) value (or a vector of values) that, in some sense, is the best representative of the fuzzy set (or fuzzy sets). This conversion is called a *defuzzification*. The defuzzified values represent actions taken by the fuzzy controller in individual control cycles.

To characterize the steps involved in designing a fuzzy controller, let us consider a very simple control problem, the problem of keeping a desired value of a single variable in spite of environmental disturbances. In this case, two conditions are usually monitored by the controller: an *error*, e, defined as the difference between the actual value of the controlled variable and its desired value, and the *derivative of the error*, $\dot{e}$, which expresses the rate of change of the error. Using values of e and $\dot{e}$, the fuzzy controller produces values of a controlling variable v, which represents relevant control actions.

Let us now discuss the basic steps involved in the design of fuzzy controllers and illustrate each of them by this simple control problem, expressed by the scheme in Fig. 12.3.

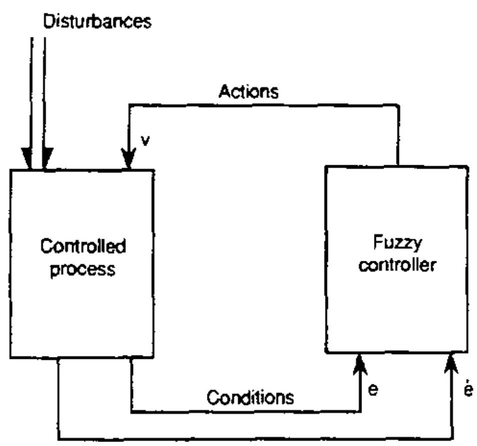

Figure 12.3 A general scheme for controlling a desired value of a single variable.

The design involves the following five steps.

Step 1. After identifying relevant input and output variables of the controller and ranges of their values, we have to select meaningful *linguistic states* for each variable and express them by appropriate fuzzy sets. In most cases, these fuzzy sets are fuzzy numbers, which represent linguistic labels such as *approximately zero*, *positive small*, *negative small*, *positive medium*, and so on.

To illustrate this step by the simple control problem depicted in Fig. 12.3, assume that the ranges of the input variables e and $\dot{e}$ are $[-a, a]$ and $[-b, b]$, respectively, and the range of the output variable v is $[-c, c]$. Assume further that the following seven linguistic states are selected for each of the three variables:

NL—*negative large*	PL—*positive large*
NM—*negative medium*	PM—*positive medium*
NS—*negative small*	PS—*positive small*
AZ—*approximately zero*	

Representing, for example, these linguistic states by triangular-shape fuzzy numbers that are equally spread over each range, we obtain the *fuzzy quantizations* exemplified for variable

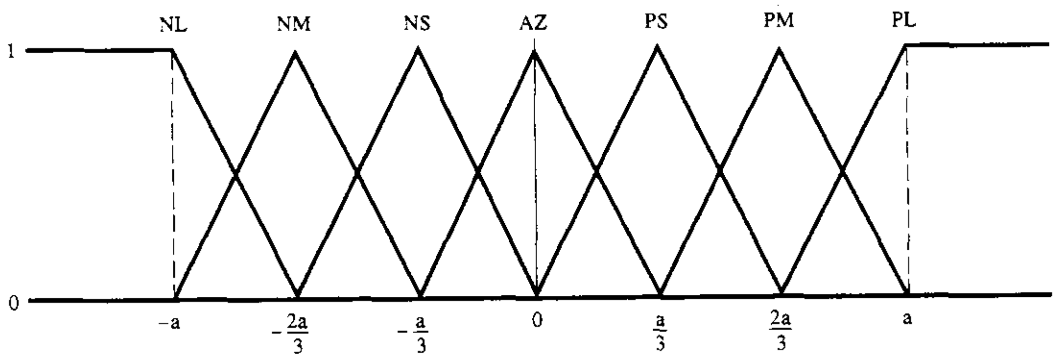

Figure 12.4 Possible fuzzy quantization of the range $[-a, a]$ by triangular-shaped fuzzy numbers.

e in Fig. 12.4; for variables $\dot{e}$ and v, value a in Fig. 12.4 is replaced with values b and c, respectively.

It is important to realize that the fuzzy quantization defined in Fig. 12.4 for the range $[-a, a]$ and the seven given linguistic labels are only a reasonable example. For various reasons, emerging from specific applications, other shapes of the membership functions might be preferable to the triangular shapes. The shapes need not be symmetric and need not be equally spread over the given ranges. Moreover, different fuzzy quantizations may be defined for different variables. Some intuitively reasonable definitions of the membership functions (e.g., those given in Fig. 12.4) are usually chosen only as preliminary candidates. They are later modified by appropriate learning methods, often implemented by neural networks.

Step 2. In this step, a *fuzzification function* is introduced for each input variable to express the associated measurement uncertainty. The purpose of the fuzzification function is to interpret measurements of input variables, each expressed by a real number, as more realistic fuzzy approximations of the respective real numbers. Consider, as an example, a fuzzification function f_e applied to variable e. Then, the fuzzification function has the form

$$f_e : [-a, a] \rightarrow \mathcal{R},$$

where $\mathcal{R}$ denotes the set of all fuzzy numbers, and $f_e(x_0)$ is a fuzzy number chosen by f_e as a fuzzy approximation of the measurement $e = x_0$. A possible definition of this fuzzy number for any $x_0 \in [-a, a]$ is given in Fig. 12.5, where ε denotes a parameter that has to be determined in the context of each particular application. It is obvious that, if desirable, other shapes of membership functions may be used to represent the fuzzy numbers $f_e(x_0)$. For each measurement $e = x_0$, the fuzzy set $f_e(x_0)$ enters into the inference process (Step 4) as a fact.

In some fuzzy controllers, input variables are not fuzzified. That is, measurements of input variables are employed in the inference process directly as facts. In these cases, function f_e has, for each measurement $e = x_0$ the special form $f_e(x_0) = x_0$.

Step 3. In this step, the knowledge pertaining to the given control problem is formulated in terms of a set of *fuzzy inference rules*. There are two principal ways in which

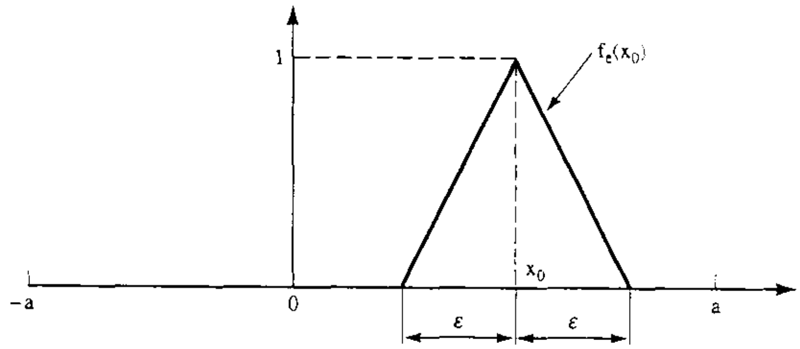

Figure 12.5 An example of the fuzzification function for variable e.

relevant inference rules can be determined. One way is to elicit them from experienced human operators. The other way is to obtain them from empirical data by suitable learning methods, usually with the help of neural networks.

In our example with variables e, $\dot{e}$, and v, the inference rules have the canonical form

$$\text{If } e = A \text{ and } \dot{e} = B, \text{ then } v = C, \tag{12.1}$$

where A, B, and C are fuzzy numbers chosen from the set of fuzzy numbers that represent the linguistic states NL, NM, NS, AZ, PS, PM, and PL. Since each input variable has, in this example, seven linguistic states, the total number of possible nonconflicting fuzzy inference rules is $7^2 = 49$. They can conveniently be represented in a matrix form, as exemplified in Fig. 12.6. This matrix and the definitions of the linguistic states (Fig. 12.4) form the fuzzy rule base of our fuzzy controller. In practice, a small subset of all possible fuzzy inference rules is often sufficient to obtain acceptable performance of the fuzzy controller. Appropriate pruning of the fuzzy rule base may be guided, for example, by statistical data regarding the utility of the individual fuzzy inference rules under specified circumstances.

<table>
<tr><td colspan="2"></td><td colspan="7">ė</td></tr>
<tr><td></td><td>v</td><td>NL</td><td>NM</td><td>NS</td><td>AZ</td><td>PS</td><td>PM</td><td>PL</td></tr>
<tr><td rowspan="7">e</td><td>NL</td><td colspan="4" rowspan="2">PL</td><td rowspan="2">PM</td><td colspan="2" rowspan="2">AZ</td></tr>
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<tr><td>AZ</td><td>PS</td><td>AZ</td><td>NS</td></tr>
<tr><td>PS</td><td>AZ</td><td>NS</td><td>NM</td></tr>
<tr><td>PM</td><td colspan="2" rowspan="2">AZ</td><td rowspan="2">NM</td><td colspan="4" rowspan="2">NL</td></tr>
<tr><td>PL</td></tr>
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Figure 12.6 An example of a fuzzy rule base.

To determine proper fuzzy inference rules experimentally, we need a set of input-output data

$$\{\langle x_k, y_k, z_k\rangle | k \in K\},$$

where z_k is a desirable value of the output variable v for given values x_k and y_k of the input variables e and $\dot{e}$, respectively, and K is an appropriate index set. Let $A(x_k), B(y_k), C(z_k)$ denote the largest membership grades in fuzzy sets representing the linguistic states of variables e, $\dot{e}$, v, respectively. Then, it is reasonable to define a *degree of relevance* of the rule (12.1) by the formula

$$i_1[i_2(A(x_k), B(y_k)), C(z_k)],$$

where i_1, i_2 are t-norms. This degree, when calculated for all rules activated by the input-output data, allows us to avoid conflicting rules in the fuzzy rule base. Among rules that conflict with one another, we select the one with the largest degree of relevance.

Step 4. Measurements of input variables of a fuzzy controller must be properly combined with relevant fuzzy information rules to make inferences regarding the output variables. This is the purpose of the *inference engine*. In designing inference engines, we can directly utilize some of the material covered in Chapters 8 and 11.

In our example with variables e, $\dot{e}$, v, we may proceed as follows. First, we convert given fuzzy inference rules of the form (12.1) into equivalent simple fuzzy conditional propositions of the form

$$\text{If } \langle e, \dot{e}\rangle \text{ is } A \times B, \text{ then } v \text{ is } C,$$

where

$$[A \times B](x, y) = \min[A(x), B(y)]$$

for all $x \in [-a, a]$ and all $y \in [-b, b]$. Similarly, we express the fuzzified input measurements $f_e(x_0)$ and $f_{\dot{e}}(y_0)$ as a single joint measurement,

$$\langle e_0, \dot{e}_0\rangle = f_e(x_0) \times f_{\dot{e}}(y_0).$$

Then, the problem of inference regarding the output variable v becomes the problem of approximate reasoning with several conditional fuzzy propositions, which is discussed in Sec. 11.4. When the fuzzy rule base consists of n fuzzy inference rules, the reasoning schema has, in our case, the form

Rule 1 :	If $\langle e, \dot{e}\rangle$ is $A_1 \times B_1$, then v is C_1
Rule 2 :	If $\langle e, \dot{e}\rangle$ is $A_2 \times B_2$, then v is C_2
....	....
Rule n :	If $\langle e, \dot{e}\rangle$ is $A_n \times B_n$, then v is C_n
Fact :	$\langle e, \dot{e}\rangle$ is $f_e(x_0) \times f_{\dot{e}}(y_0)$
Conclusion :	v is C

The symbols $A_j, B_j, C_j (j = 1, 2, \ldots, n)$ denote fuzzy sets that represent the linguistic states of variables $e, \dot{e}, v$, respectively.

For each rule in the fuzzy rule base, there is a corresponding relation R_j, which is determined as explained in Sec. 8.3. Since the rules are interpreted as disjunctive, we may use (11.25) to conclude that the state of variable v is characterized by the fuzzy set

$$C = \bigcup_j [f_e(x_0) \times f_{\dot{e}}(y_0)] \overset{i}{\circ} R_j, \tag{12.2}$$

where $\overset{i}{\circ}$ is the sup-i composition for a t-norm i. The choice of the t-norm is a matter similar to the choice of fuzzy sets for given linguistic labels. The t-norm can be either elicited from domain experts or determined from empirical data.

The method of interpolation explained in Sec. 11.4 and illustrated in Fig. 11.3 is the usual method employed in simple fuzzy controllers for determining the resulting fuzzy set C in the described reasoning schema. The formulation of the method in Sec. 11.4 can be applied to our case by taking $\mathcal{X} = \langle e, \dot{e} \rangle$ and $\mathcal{Y} = v$. Linguistic states of $\langle e, \dot{e} \rangle$ are fuzzy sets defined on $[-a, a] \times [-b, b]$, which are calculated from their counterparts for e and $\dot{e}$ by the minimum operator.

Step 5. In this last step of the design process, the designer of a fuzzy controller must select a suitable *defuzzification method.* The purpose of defuzzification is to convert each conclusion obtained by the inference engine, which is expressed in terms of a fuzzy set, to a single real number. The resulting number, which defines the action taken by the fuzzy controller, is not arbitrary. It must, in some sense, summarize the elastic constraint imposed on possible values of the output variable by the fuzzy set. The set to be defuzzified in our example is, for any input measurements $e = x_0$ and $\dot{e} = y_0$, the set C defined by (12.2).

A number of defuzzification methods leading to distinct results were proposed in the literature. Each method is based on some rationale. The following three defuzzification methods have been predominant in the literature on fuzzy control.

Center of Area Method

In this method, which is sometimes called the *center of gravity method* or *centroid method*, the defuzzified value, $d_{CA}(C)$, is defined as the value within the range of variable v for which the area under the graph of membership function C is divided into two equal subareas. This value is calculated by the formula

$$d_{CA}(C) = \frac{\int_{-c}^{c} C(z)z\,dz}{\int_{-c}^{c} C(z)\,dz}. \tag{12.3}$$

For the discrete case, in which C is defined on a finite universal set $\{z_1, z_2, \ldots, z_n\}$, the formula is

$$d_{CA}(C) = \frac{\sum_{k=1}^{n} C(z_k)z_k}{\sum_{k=1}^{n} C(z_k)}. \tag{12.4}$$

If $d_{CA}(C)$ is not equal to any value in the universal set, we take the value closest to it.

Observe that the values

$$\frac{C(z_k)}{\sum_{k=1}^{n} C(z_k)}$$

for all $k = 1, 2, \ldots, n$ form a probability distribution obtained from the membership function C by the ratio-scale transformation. Consequently, the defuzzified value $d_{CA}(C)$ obtained by formula (12.4) can be interpreted as an expected value of variable v.

Center of Maxima Method

In this method, the defuzzified value, $d_{CM}(C)$, is defined as the average of the smallest value and the largest value of v for which $C(z)$ is the height, $h(C)$, of C. Formally,

$$d_{CM}(C) = \frac{\inf M + \sup M}{2}, \tag{12.5}$$

where

$$M = \{z \in [-c, c] | C(z) = h(C)\}. \tag{12.6}$$

For the discrete case,

$$d_{CM}(C) = \frac{\min\{z_k | z_k \in M\} + \max\{z_k | z_k \in M\}}{2}, \tag{12.7}$$

where

$$M = \{z_k | C(z_k) = h(C)\}. \tag{12.8}$$

Mean of Maxima Method

In this method, which is usually defined only for the discrete case, the defuzzified value, $d_{MM}(C)$, is the average of all values in the crisp set M defined by (12.8). That is,

$$d_{MM}(C) = \frac{\sum_{z_k \in M} z_k}{|M|}. \tag{12.9}$$

In the continuous case, when M is given by (12.6), $d_{MM}(C)$ may be defined as the arithmetic average of mean values of all intervals contained in M, including intervals of length zero. Alternatively, $d_{MM}(C)$ may be defined as a weighted average of mean values of the intervals, in which the weights are interpreted as the relative lengths of the intervals.

An application of the four defuzzification methods (d_{CA}, d_{CM}, d_{MM}, and d_{MM} weighted) to a particular fuzzy set is illustrated in Fig. 12.7.

It is now increasingly recognized that these defuzzification methods, as well as other methods proposed in the literature, may be viewed as special members of parametrized families of defuzzification methods. For the discrete case, an interesting family is defined by the formula

$$d_p(C) = \frac{\sum_{k=1}^{n} C^p(z_k) z_k}{\sum_{k=1}^{n} C^p(z_k)}, \tag{12.10}$$

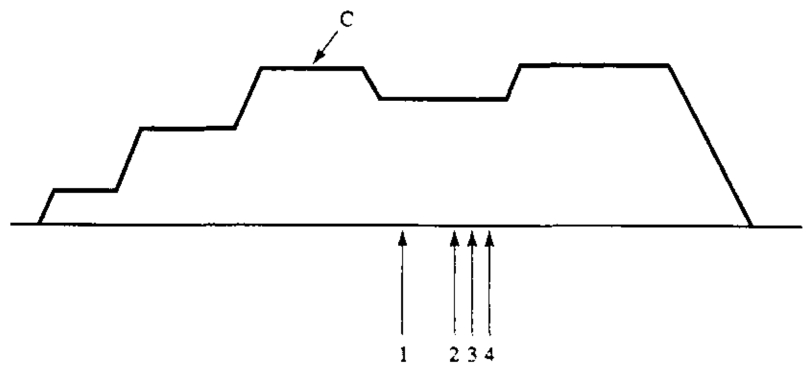

Figure 12.7 Illustration of the described defuzzification methods: $1 - d_{CA}$, $2 - d_{MM}$, $3 - d_{CM}$, $4 - d_{MM}$ (weighted).

where $p \in (0, \infty)$ is a parameter by which different defuzzification methods are distinguished. This parameter has an interesting interpretation. When $p = 1$, the center of area method is obtained. When $p \neq 1$, the parameter introduces a bias into the probability distribution obtained from C by the ratio-scale transformation. When $p < 1$, probabilities of values z_k for which $C(z_k) < h(C)$ are magnified. The magnification increases with decreasing value p and, for each given value of p, it increases with decreasing value of $C(z_k)$. When $p > 1$, the biasing effect is inversed; that is, probabilities of value z_k for which $C(z_k) < h(C)$ are reduced. The reduction increases with decreasing values $C(z_k)$ and increasing values p.

When $p \to 0$, all values in $\{z_1, z_2, \ldots, z_n\}$ are given equal probabilities and, hence, $d_0(C)$ is equal to their arithmetic average. In this case, the shape of the membership function C is totally discounted. Taking $d_0(C)$ as a value that represents a summary of fuzzy set C may be interpreted as an expression of very low confidence in the inference process. At the other extreme, when $p \to \infty$, we obtain the mean of maxima method, and taking $d_\infty(C) = d_{MM}(C)$ may be interpreted as the expression of full confidence in the reasoning process. Hence, the value of p of the chosen defuzzification may be interpreted as an indicator of the confidence placed by the designer in the reasoning process.

Observe that one particular value of p is obtained by the uncertainty invariance principle explained in Sec. 9.7. In this case, the probability distribution that is used for calculating the expected value of variable v preserves all information contained in the fuzzy set C. Hence, the defuzzification method based on this value is the only method fully justified in information-theoretic terms.

For the continuous case, the following formula is a counterpart of (12.10):

$$d_p(C) = \frac{\int_{-c}^{c} C^p(z) z\, dz}{\int_{-c}^{c} C^p(z)\, dz}. \tag{12.11}$$

The formula is not directly applicable for $p \to \infty$, but it is reasonable (in analogy with the discrete case) to define d_∞ as d_{MM} given by (12.9).

12.3 FUZZY CONTROLLERS: AN EXAMPLE

In this section, we describe a particular fuzzy controller of the simple type characterized in Fig. 12.3. We use only one example since descriptions of many other fuzzy controllers, designed for a great variety of control problems, can readily be found in the literature (Note 12.2).

We chose to describe a fuzzy controller whose control problem is to stabilize an inverted pendulum. This control problem, which has been quite popular in both classical control and fuzzy control, has a pedagogical value. It can easily be understood, due to its appeal to common sense, and yet it is not overly simple. We describe a very simple version, which consists of only seven fuzzy inference rules. It was designed and implemented by Yamakawa [1989]. He demonstrated that even such a simple fuzzy controller works reasonably well for poles that are not too short or too light, and under environmental disturbances that are not too severe. Although performance can be greatly improved by enlarging the fuzzy rule base, the simple version is preferable from the pedagogical point of view.

The problem of stabilizing an inverted pendulum, which is illustrated in Fig. 12.8, is described as follows. A movable pole is attached to a vehicle through a pivot, as shown in the figure. This situation can be interpreted as an inverted pendulum. The control problem is to keep the pole (pendulum) in the vertical position by moving the vehicle appropriately. The three variables involved in this control problem have the following meaning: e is the angle between the actual position of the pole and its desirable vertical position, $\dot{e}$ is the derivative (rate of change) of variable e, and v is proportional to the velocity, $\dot{w}$, of the vehicle. While variable e is directly measured by an appropriate angle sensor, its derivative $\dot{e}$ is calculated from successive measurements of e. When the pole is tilted toward left (or toward right), e is viewed as negative (or positive, respectively), and a similar convention applies to $\dot{e}$. Variable v is a suitable electrical quantity (electric current or voltage). Its values determine, through an electric motor driven by an appropriate servomechanism, the force applied to the vehicle. Again, the force is viewed as negative (or positive) when it causes the vehicle to move to the left (or to the right, respectively). For convenience, we may express v in a suitable scale for which it is numerically equal to the resulting force. Then, propositions about v may be directly interpreted in terms of the force applied to the vehicle.

Observe that this simplified formulation of the control problem does not include the requirement that the position of the vehicle also be stabilized. Two additional input variables would have to be included in the fuzzy controller to deal with this requirement: a variable defined by the distance (positive or negative) between the actual position of the vehicle and its desirable position, and the derivative of this variable expressing the velocity of the vehicle (positive or negative). Assuming that seven linguistic states were again recognized for each of the variables, the total number of possible nonconflicting fuzzy inference rules would become $7^4 = 2,401$. It turns out from experience that only a small fraction of these rules, say 20 or so, is sufficient to achieve a high performance of the resulting fuzzy controller.

To compare fuzzy control with classical control in dealing with the problem of stabilizing an inverted pendulum, we briefly describe a mathematical model of the mechanics involved by which proper movements of the vehicle would be determined in a classical controller. The model consists of a system of four differential equations, whose derivation is outside the scope of this text. The equations are

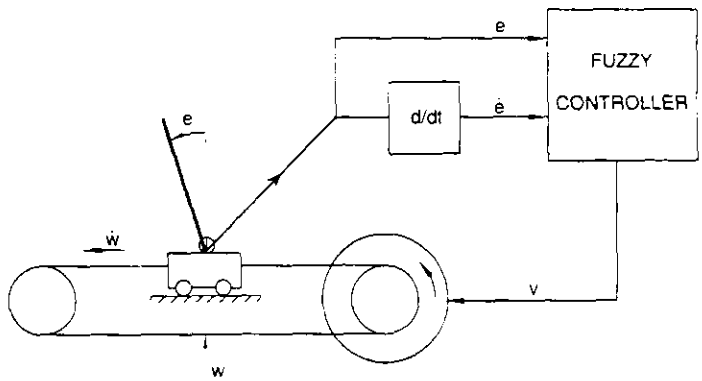

Figure 12.8 Fuzzy controller as a stabilizer of an inverted pendulum.

$$I\ddot{e} = VL\sin e - HL\cos e,$$
$$V - mg = -mL(\ddot{e}\sin e + \dot{e}^2\cos e),$$
$$H = m\ddot{w} + mL(\ddot{e}\cos e - \dot{e}^2\sin e),$$
$$U - H = M\ddot{w},$$

where $e, \dot{e}, \ddot{e}$ are the angle and its first and second derivatives, $2L$ is the length of the pendulum, $\ddot{w}$ is the second derivative of the position of the vehicle, m and M are masses of the pendulum and the vehicle, H and V are horizontal and vertical forces at the pivot, V is the driving force given to the vehicle, and $I = mL^2/3$ is the moment of inertia. It is clearly difficult to comprehend this model intuitively. Moreover, since the first three equations are nonlinear, the system is difficult, if not impossible, to solve analytically. Hence, we have to resort to computer simulation, which is not suitable for real-time control due to excessive computational demands. To overcome these difficulties, special assumptions are usually introduced to make the model linear. This, however, restricts the capabilities of the controller. Another difficulty of the model-based classical control is that the model must be modified whenever some parameters change, (e.g., when one pendulum is replaced with another of different size and weight). Fuzzy control, based on intuitively understandable linguistic control rules, is not subject to these difficulties.

Let us now discuss the main issues involved in the design of a fuzzy controller for stabilizing an inverted pendulum. The discussion is organized in terms of the five design steps described in Sec 12.2.

Step 1. It is typical to use seven linguistic states for each variable in fuzzy controllers designed for simple control problems such as the stabilization of an inverted pendulum. The linguistic states are usually represented by fuzzy sets with triangular membership functions, such as those defined in Fig. 12.4. Let us choose this representation for all three variables, each defined for the appropriate range.

Step 2. Assume, for the sake of simplicity, that measurements of input variables are employed directly in the inference process. However, the distinction made in the inference process by fuzzified inputs is illustrated in Fig. 12.11.

Step 3. Following Yamakawa [1989], we select the seven linguistic inference rules defined by the matrix in Fig. 12.9. The complete representation of these linguistic rules by the fuzzy sets chosen in Step 1 is shown in Fig. 12.10. Observe that the linguistic states NL and PL do not even participate in this very restricted set of inference rules.

$\dot{e}$ \ e	NM	NS	AZ	PS	PM
NS		NS		AZ	
AZ	NM		AZ		PM
PS		AZ		PS	

Figure 12.9 Minimum set of linguistic inference rules to stabilize an inverted pendulum.

The inference rules can easily be understood intuitively. For example, if the angle is negative small ($e = NS$) and its rate of change is negative small ($\dot{e} = NS$), then it is quite natural that the velocity of the vehicle should be also negative small ($v = NS$) to make a correction in the way the pole is moving. On the other hand, when the angle is positive small ($e = PS$) and its rate of change is negative small ($\dot{e} = NS$), the movement of the pole is self-correcting; consequently, the velocity of the vehicle should be approximately zero ($v = AZ$). Other rules can be easily explained by similar common-sense reasoning.

Step 4. To illustrate the inference engine, let us choose the interpolation method explained in Sec. 11.4 and extended as explained in Sec. 12.2. This method is frequently used in simple fuzzy controllers. For each pair of input measurements, $e = x_0$ and $\dot{e} = y_0$, or their fuzzified counterparts, $e = f_e(x_0)$ and $\dot{e} = f_{\dot{e}}(y_0)$, we first calculate the degree of their compatibility $r_j(x_0, y_0)$ with the antecedent (a fuzzy number) of each inference rule j. When $r_j(x_0, y_0) > 0$, we say that rule j fires for the measurements. We can see by a careful inspection of our fuzzy rule base, as depicted in Fig. 12.10, that: (a) at least one rule fires for all possible input measurements, crisp or fuzzified; (b) no more that two rules can fire when the measurements of input variables are crisp; (c) more than two rules (and perhaps as many as five) can fire when the measurements are fuzzified. For each pair of input measurements, the value of the output variable v is approximated by a fuzzy set $C(z)$ determined by (12.2), as illustrated in Fig. 12.11 for both crisp and fuzzified measurements. Observe that only rules 1 and 3 fire for the given crisp input measurements, while rules 1, 2, and 3 fire for their fuzzified counterparts.

Step 5. The most frequently used defuzzifications method in the simple fuzzy controller is the centroid method. If we accept it for our controller, we obtain our defuzzified values by (12.3), as exemplified in Fig. 12.11.

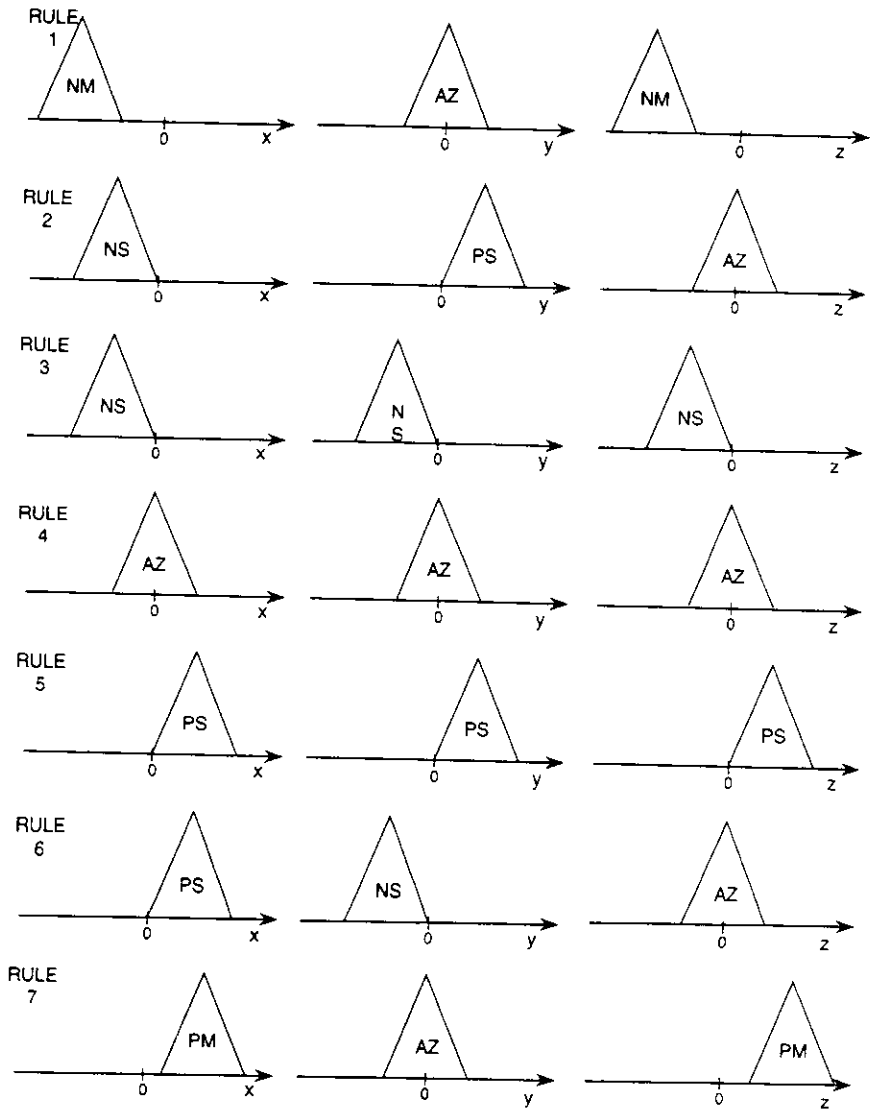

Figure 12.10 Fuzzy rule base of the described fuzzy controller designed for stabilizing an inverted pendulum.

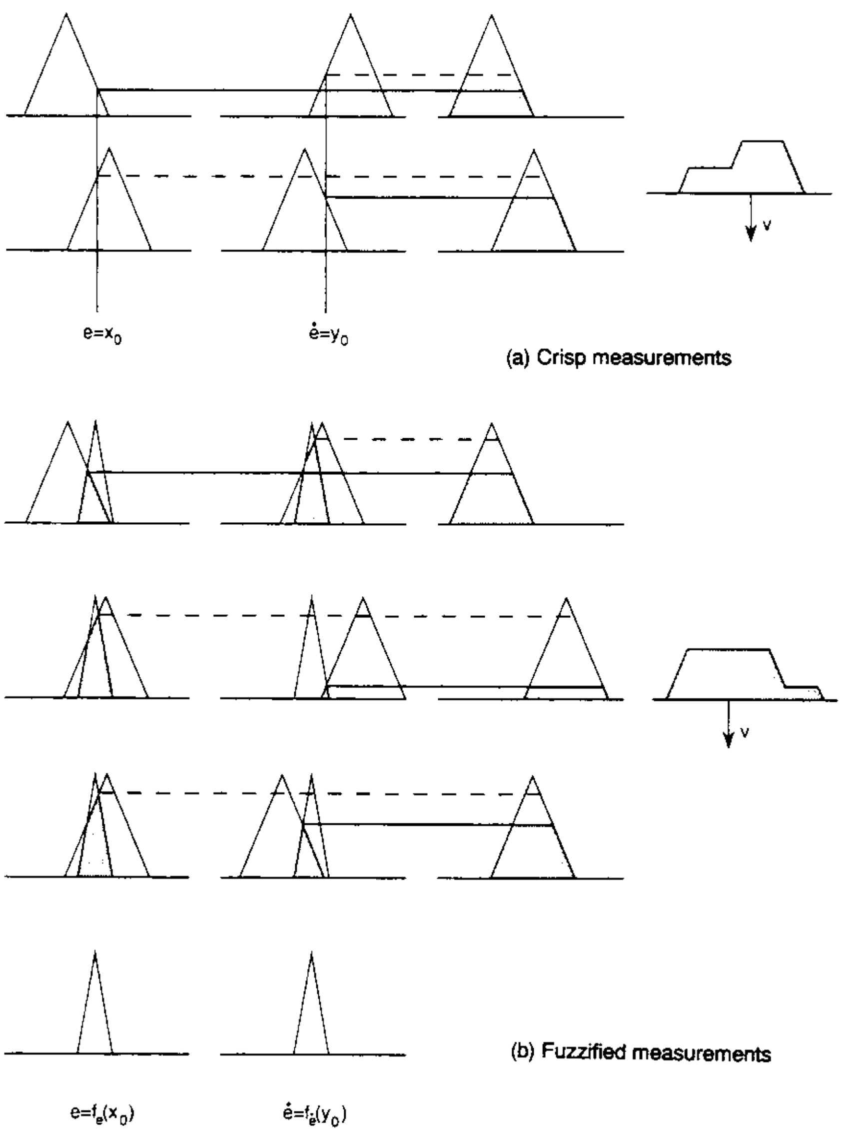

Figure 12.11 Examples of fuzzy inference rules (of the fuzzy rule base specified in Fig. 11.10) that fire for given measurements of input variables: (a) crisp measurements; (b) fuzzified measurements.